INDEX THEORY FOR STRATEGIC-FORM GAMES WITH AN APPLICATION TO EXTENSIVE-FORM GAMES

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Abstract. Whenever equivalent mixed strategies of a player are identified (topologically) in a normal-form game, the resulting space may not be a simplex anymore but is a general polytope. We show that an index/degree theory of equilibria can be developed in full generality for games in which the strategy sets of the players are general polytopes and their payoff functions are multiaffine. Index and degree theories work as a tool that helps identify equilibria that are robust to payoff perturbations of the game. Because the strategy set of each player is the result of the identification of equivalent mixed strategies, the resulting polytope is of lower dimension than the original mixed strategy simplices. This, together with an index theory, has algorithmic applications for checking for robustness of equilibria as well as finding equilibria in extensive-form games.

1. Introduction

Strategic-form games are games where the strategy set of the players are polytopes and their payoff functions are multiaffine[1] in the product of these polytopes[2]. They arise naturally in the analysis of (finite) normal-form games when equivalent mixed strategies of a player are identified. The principle of *Invariance* is a decision-theoretic principle introduced by Kohlberg and Mertens (1986) that requires a given solution of a game to be a solution of its *reduced normal-form*, that is, the game obtained by eliminating all pure strategies that are convex combinations of other pure strategies[3]. The reason for the adoption of Invariance is that since there is no strategic difference between equivalent pure strategies, they should be treated as equal. The same reasoning can be invoked for the deletion of equivalent mixed-strategies of a finite game. The consequence of such deletion, however, is that the resulting strategy space is not necessarily a simplex anymore, but is a polytope.[4]

The deletion of redundant mixed-strategies allows the representation of a finite game in a similar manner to the normal-form, using now polytopes as strategy sets and multiaffine functions as payoff functions for the players. The strategic-form representation of a finite game is therefore an alternative, more concise representation of a finite game, which eliminates some information that is present in the normal-form. The strategic-form preserves Nash-equilibria from the normal-form but it is not clear whether more complex equilibrium refinement concepts are also preserved.

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[1] "Multiaffine" will be understood as affine in each coordinate.

[2] In strategic-form games, the strategy set of a player does not have to be a simplex: it can be a general polytope.

[3] Several pervasive solution concepts in Game Theory – such as sequential equilibrium – fail to be invariant. See Kohlberg and Mertens (1986).

[4] Mertens (2004) and Von Stengel (1996) propose the identification of equivalent mixed-strategies, resulting in polytopes as strategy sets. Mertens coined the name Strategic-form games, which we use in this paper.

Several equilibrium refinement concepts widely used in the literature – such as sequential, proper, perfect equilibria etc. – require some sort of robustness[5] of equilibria to perturbations of the game. These perturbations are usually of two types: strategy set perturbations or payoff perturbations. But even strategy set perturbations – such as the ones required in the case of perfect, proper or sequential equilibria – imply a particular type of payoff perturbation. So if in principle we can check whether an equilibrium is robust to general payoff perturbations, this implies that this equilibrium is going to satisfy several refinement concepts simultaneously. Index and degree theories of normal-form games (see Govindan and Wilson (2005)) are tools designed to identify which equilibria are robust to payoff perturbations. In this paper we are going to develop index and degree theories for strategic-form games and show their equivalence to the normal-form theory, thereby showing that *no relevant information to robustness to payoff perturbation is lost by restricting the analysis to the strategic-form representation of a finite game*. This is not a simple matter: once redundant strategies of players are eliminated from the normal-form of a game, the dimension of the payoff space of the players decrease. It is not clear *a priori* that in the game resulting from the elimination of redundant strategies, the equilibria identified as robust by the strategic-form index theory will also remain robust in the original normal-form game: the space of perturbations of the latter is typically much larger than the former. By showing the equivalence between strategic-form and normal-form index theories, we show that indeed no information regarding robustness to payoff perturbation is lost by restricting to the strategic-form.

A byproduct of such restriction is that the strategic-form polytopes obtained by identifying redundant mixed strategies are of lower dimension than the normal-form mixed strategy sets. This will imply a significant gain in computational complexity for algorithms dedicated to find equilibria and also check whether the equilibria are robust. The gain is particularly significant for extensive-form games.

We then proceed to apply the degree and index theories to extensive-form games. First, we represent an extensive-form game in strategic-form using *Enabling Strategies* as the polytope for each player. This representation is called the *strategic-form of the extensive-form game*. An immediate gain from considering the strategic-form representation of an extensive-form game is that the dimension of the set of Enabling Strategies grows linearly with respect to the number of terminal nodes of a game tree, whereas the number of pure strategies in the normal-form can grow exponentially. Together with a well defined theory of index/degree of equilibria for such games, this implies a faster implementation of algorithms dedicated to find equilibria, such as the ones proposed in Govindan and Wilson (2003). The strategic-form of the extensive-form also maintains convexity and compactness of the strategy set as well as multilinearity of the payoff function of each player – which are convenient properties for the application of algorithms[6]. The strategic-form of an extensive-form game is essentially equivalent to the *Sequence-form* proposed by Von Stengel (1996). Enabling Strategies were first presented in Govindan and Wilson (2002) and we use their formulation in our analysis.

The strategic-form representation of a finite game will also serve the purpose of establishing the link between two previously unrelated theories. In Govindan and Wilson (2002) a Structure Theorem for extensive-form games is proved using a "perturbed version" of the Enabling strategy polytope as the strategy set of

[5]An equilibrium is "robust" if for sufficiently small perturbations of the game, there are equilibria close by.

[6]Algorithms for finding equilibria are usually applied to programs that require some form of linearity of the objective function and convexity of the constraint set – the solution of the program is an equilibrium of the game. In the zero-sum case, the program is an LP (linear program). In the general two-player case, it is an LCP (linear complementary problem) and in the N-player case, it is a multilinear optimization problem. See Von Stengel (1996) or Nisan et al. (2007).

each player. This Structure Theorem for extensive-form games – as it happens with the normal-form case – defines a theory of degree of equilibria for extensive-form games. However the relation between the degrees of equilibria in the Govindan and Wilson degree theory for extensive-form games and the normal-form degree theory is not known. We establish the precise relation between the two theories in this paper. The relation shows how the Govindan and Wilson degree theory can essentially be used as an "approximation" for the the normal-form degree theory. Ultimately, our result (Theorem 6.19) establishes that the two theories, although formulated in very different environments, capture the same robustness properties of equilibria.

Most of the technical proofs are left to the Appendix. The ones that highlight important features and can be executed avoiding technical machinery are kept.

1.1. **Organization and Results.**

(1) Section 2 presents the strategic-form representation of a finite game.
(2) Section 3 develops index and degree theories for strategic-form games.
(3) Section 4 contains the first main result of the paper: the different index and degree theories of equilibria in strategic-form games are all equivalent - which parallels what happens for normal-form games. This is Theorem 4.1.
(4) Section 5 contains the second main result of the paper: no fundamental information is lost – regarding robustness of equilibria to payoff perturbations – when the analysis is restricted to the strategic-form of the game. This is Theorem 5.1 and Corollary 5.2.
(5) Section 6 presents the strategic-form of an extensive-form game and discusses its computational advantages compared to the normal-form representation of an extensive-form game. Also in this section is the third main result (Theorem 6.19), presenting the relation between Govindan and Wilson degree theory, normal-form degree theory and strategic-form degree theory.

2. Strategic-form Games

Definition 2.1. Let $G = (N, (\Sigma_n)_{n\in N}, (G_n)_{n\in N})$ be a finite normal-form game, where $N := \{1, ..., N\}$, $\Sigma_n = \Delta(S_n)$ is the set of mixed strategies of player n, S_n the (finite) set of pure strategies of player n and G_n the payoff function to player n. Two mixed strategies τ_n and τ'_n are *equivalent* if for each player $j \in N, s_{-n} \in S_{-n}$,

$$G_j(\tau_n, s_{-n}) = G_j(\tau'_n, s_{-n})$$

We construct the *strategic-form of a normal-form game*, which is obtained through the identification of equivalent mixed strategies of a player.

Each payoff function G_n can be uniquely extended to a multilinear function on $\times_n \mathbb{R}^{|S_n|}$, thus we still denote by G_n the extension. Consider the following equivalence relation on $\mathbb{R}^{|S_n|}$: $x_n \sim y_n$, for $x_n, y_n \in \mathbb{R}^{|S_n|} \Leftrightarrow \forall j \in N, \forall s_{-n} \in S_{-n}, G_j(x_n, s_{-n}) = G_j(y_n, s_{-n})$. Let $\mathbb{R}^{|S_n|}/\sim$ be the set of equivalence classes given by equivalence relation $\sim$. We endow this set with the quotient topology given by this equivalence relation. The quotient space $\mathbb{R}^{|S_n|}/\sim$ can be given a real vector space structure from operations in $\mathbb{R}^{|S_n|}$: for addition, if $[x_n], [y_n] \in \mathbb{R}^{|S_n}/\sim$, define $[x_n] \bigoplus [y_n] := [x_n + y_n]$; for scalar multiplication, let $\alpha \in \mathbb{R}$ and define $\alpha[x_n] := [\alpha x_n]$. Because $\mathbb{R}^{|S_n|}$ is finite dimensional, so is $\mathbb{R}^{|S_n|}/\sim$. Therefore, $\mathbb{R}^{|S_n|}/\sim$ is a Euclidean space and we denote it by $\mathbb{R}^{k_n} := \mathbb{R}^{|S_n|}/\sim$.

FIGURE 1. Extensive-form game G

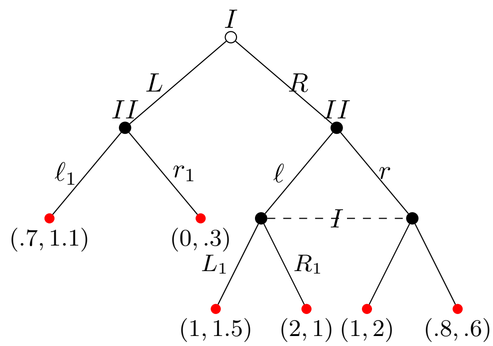


TABLE 1. Normal-form game of G

		II			
		$\ell_1\ell$	$\ell_1 r$	$r_1\ell$	$r_1 r$
	LL_1	.7, 1.1	.7, 1.1	0, .3	0, .3
I	LR_1	.7, 1.1	.7, 1.1	0, .3	0, .3
	RL_1	1, 1.5	1, 2	1, 1.5	1, 2
	RR_1	2, 1	.8, .6	2, 1	.8, .6

Consider now the partition mapping $\Pi_n : \mathbb{R}^{|S_n|} \to \mathbb{R}^{k_n}$ given by $x_n \mapsto [x_n]$ and its restriction to Σ_n given by $(\Pi_n)_{|\Sigma_n}$. It follows from the definition that $(\Pi_n)_{|\Sigma_n}$ is an affine mapping which implies that $(\Pi_n)_{|\Sigma_n}(\Sigma_n)$ is a polytope.

For each $n \in N$, let $P_n := (\Pi_n)_{|\Sigma_n}(\Sigma_n)$ and define $V_n : \times_{n\in N} P_n \to \mathbb{R}$ by letting $V_n(v_1, ..., v_n) := G_n(s_1, ..., s_n)$, where for each $n \in N$, v_n is a vertex of P_n and $\Pi_n(s_n) = v_n$, and then extending it affinely in each coordinate. This definition implies $V_n \circ (\times_j \Pi_j) = G_n$. The strategic-form game defined by $(N, (P_n)_{n\in N}, (V_n)_{n\in N})$ is a *strategic-form representation* of the normal-form game $G = (N, (\Sigma_n)_{n\in N}, (G_n)_{n\in N})$. Any polytope Y_n which is affinely bijective with P_n can be used as a strategy set for player n in an alternative strategic-form representation of the normal-form game G: let $e_n : Y_n \to P_n$ be an affine bijection. Then define $V'_n : \times_n Y_n \to \mathbb{R}$ as $V'_n := V_n \circ (\times_n e_n)$. The strategic-form game $(N, (Y_n)_{n\in N}, (V'_n)_{n\in N})$ is also a strategic-form of the normal-form game G.

In practical examples, some strategic-form representations might be easier to compute than others. In the case of extensive-form games we will focus in Section 6 on constructing a specific type of strategic-form representation, where the strategy sets and payoffs of the players can be derived directly from the data defining the game-tree. In extensive-form games, the number of pure strategies of a player is typically exponential in the number of terminal nodes of the game-tree, so deriving the strategic-form from the game-tree directly without having to compute the normal-form is preferable.

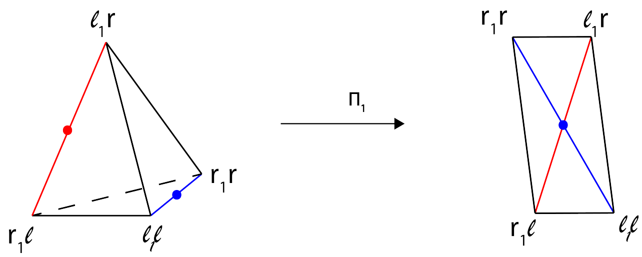
Example 2.2. In Figure 1 we depict an extensive-form game G and show that the strategy polytope of player II obtained from the identification of his equivalent mixed-strategies is not a simplex.

We can identify the pure strategies LL_1 with LR_1 of player I, since they are equivalent, which gives us the reduced normal-form of Table 2. In the reduced normal-form, the strategy sets of the players are still

Table 2. Reduced normal-form game of G

		II			
		$\ell_1\ell$	$\ell_1 r$	$r_1\ell$	$r_1 r$
	LR_1	$.7, 1.1$	$.7, 1.1$	$0, .3$	$0, .3$
I	RL_1	$1, 1.5$	$1, 2$	$1, 1.5$	$1, 2$
	RR_1	$2, 1$	$.8, .6$	$2, 1$	$.8, .6$

Figure 2. Identification



simplices. Now, notice that the equal mixture of pure strategies $\ell_1 r$ and $r_1\ell$ of player II is equivalent to the equal mixture of $\ell_1\ell$ and $r_1 r$. When identified, these mixed strategies give rise to a "square", as illustrated in Figure 2.

3. Strategic-form Index and Degree Theories

3.1. **Strategic-form Degree.** A strategic-form game has in general finitely many equilibrium (connected) components.[7] The concept of degree of an equilibrium component is a tool that helps identifying which of the equilibrium components of the game are robust to payoff perturbations in the sense that for games nearby there will be equilibria nearby the component. In practice the degree of a component of equilibria C of a game V is an integer number; if this number is different from 0, the component is guaranteed to be robust to payoff perturbations.

In order to define the concept of degree it is first necessary to prove that the Nash equilibrium graph for strategic-form games possesses certain geometric properties. This is the content of Theorem 3.1 below.

Fix $N = \{1, ..., N\}$ a set of players and a polytope $P_n \subset \mathbb{R}^{m_n}$ for each player $n \in N$ with $P := \times_n P_n$. Let $\mathcal{E}^* := \{(V, p) \in \Pi_n A(P) \times P | \ p \text{ is an equilibrium of } V\}$, where $A(P)$ is the M-dimensional linear space of multiaffine functions from P to $\mathbb{R}$ – where the linear space structure is given by pointwise addition and scalar multiplication. The linear space $\Pi_n A(P)$ is a NM-dimensional Euclidean space and we denote its one-point compactification by $\overline{\Pi_n A(P)}$. Recall that the one-point compactification $\overline{\Pi_n A(P)}$ is homeomorphic to the sphere $\mathbb{S}^{NM}$.

Theorem 3.1 shows that $\mathcal{E}^*$ is a topological manifold homeomorphic to the space of payoffs $\prod_n A(P)$ and can be given an orientation[8], through the homeomorphism, from the orientation of the Euclidean space

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[7]Just like it happens for normal-form games, given a strategic-form game V, the set of its equilibria is semi-algebraic and has therefore finitely many connected components.

[8]An *orientation* for $\bar{\mathcal{E}}^*$ is simply a generator of the homology group $H_{NM}(\bar{\mathcal{E}}^*)$. Since the homomorphism $\bar{h}_*$ is an isomorphism, if $\mu \in H_{NM}(\mathbb{S}^{NM})$ is a generator, then $\bar{h}_*^{-1}(\mu)$ is a generator of $H_{NM}(\bar{\mathcal{E}}^*)$.

$\Pi_n A(P)$. Let $\text{proj}^* : \mathcal{E}^* \to \Pi_n A(P)$ be defined by $\text{proj}^*(V, p) = V$. Theorem 3.1 also shows that $\text{proj}^* \circ (\theta^*)^{-1}$ has the same homotopy type of the identity map on $\Pi_n A(P)$.

Theorem 3.1. *There exists a homeomorphism $\theta^* : \mathcal{E}^* \to \Pi_n A(P)$ such that there is a linear homotopy between $\text{proj}^* \circ (\theta^*)^{-1}$ and the identity function on $\Pi_n A(P)$ and this homotopy extends to a homotopy on the one-point compactification of $\Pi_n A(P)$.*

Proof. See Appendix, subsection 7.1. ◻

Let $(V, C) \in \mathcal{E}^*$ be such that C is a connected component of equilibria of the game with payoff functions $V \in \Pi_n A(P)$. Let $U \subset \mathcal{E}^*$ be an open neighborhood of (V, C) whose closure contains no pair (V, p) not already in (V, C). The *local degree of $\text{proj}|_U : U \to \Pi_n A(P)$ over V* is the integer $\deg_V(\text{proj}|_U)$ that defines the following homomorphism in singular homology:

$$(\text{proj}|_U)_* : H_{NM}(U, U - \{(V, C)\}) \to H_{NM}(\overline{\Pi_n A(P)}), \overline{\Pi_n A(P)} - \{V\}),$$

where $H_{NM}(U, U - \{(V, C)\})$ is oriented according to the following composition of homomorphisms:

$$\mathbb{Z} = H_{NM}(\overline{\Pi_n A(P)}) \to H_{NM}(\overline{\Pi_n A(P)}, \overline{\Pi_n A(P)} - K) \to H_{NM}(W, W - K) \to H_{NM}(U, U - \{(V, C)\}),$$

where $K = \theta(V, C)$ and $W = \theta(U)$; the first and second arrows correspond to inclusion, where the second is an isomorphism by excision, and the third is the isomorphism $(\theta)_*^{-1}$.

Definition 3.2. Let V be a strategic-form game and C be a component of equilibria of V. Let $U \subset \mathcal{E}^*$ be an open neighborhood of (V, C) such that $[(V, p) \in \text{cl}(U) \Rightarrow p \in C]$. Then the degree of C, denoted $\deg_V(C)$, is defined as $\deg_V(\text{proj}|_U)$.

Remark 3.3. We call a neighborhood $U \subset \mathcal{E}^*$ of Definition 3.2 *admissible for C*.

The next result shows that the degree of an equilibrium component is invariant under any representation of a strategic-form game.

Proposition 3.4. *Let P_n and P'_n be two polytopes and $E_n : P_n \to P'_n$ an affine bijection with $E := \times_n E_n$. Let $P := \times_n P_n$ and $P' = \times_n P'_n$. Let C be an equilibrium component of the strategic-form game $V = (N, (P_n)_{n \in N}, (V_n)_{n \in N})$. Then $E(C)$ is an equilibrium component of $V' = (N, (P'_n)_{n \in N}, (V'_n)_{n \in N})$, where $V'_n = V_n \circ E^{-1}$ and $\deg_V(C) = \deg_{V'}(E(C))$.*

Proof. See Appendix, subsection 7.1 ◻

The next proposition shows that an equilibrium component with degree nonzero is guaranteed to be robust to perturbation of payoffs. It follows from a simple topological property of the map proj^* and from a straightforward application of a known result in Algebraic Topology.

Proposition 3.5. *Let C be an equilibrium component of the strategic-form game $V = (N, (P_n)_{n \in N}, (V_n)_{n \in N})$. Assume $\deg_V(C) \neq 0$ and let U be admissible for C. Then there exists a neighborhood $W \subset \Pi_n A(P)$ of $V = (V_n)_{n \in N}$ such that for any $V' \in W$, there exists an equilibrium p' of V' with $(V', p') \in U$.*

Proof. See Appendix, subsection 7.1 ◻

3.2. **Strategic-form Index Theory.**

3.2.1. *Nash-map Index Theory.* An alternative to checking the robustness of equilibria through degrees is to use the fixed point index. This idea involves treating equilibria as fixed points of certain maps and then checking whether the fixed points are robust to perturbations of the maps. We formalize this idea below.

Definition 3.6. A Nash-map is a continuous function $f : \Pi_n A(P) \times P \to P$ such that for each $V \in \Pi_n A(P)$, the fixed points of its restriction f_V to $\{V\} \times P$, viewed as a map from P to itself, are the Nash equilibria of V.

Definition 3.7. Fix $C \subset W \subset \mathbb{R}^m$ where C is compact and W is open. Recall the one-point compactification $\mathbb{R}^m$ is an m-dimensional sphere. The fundamental class $\mathcal{O}_C \in H_m(W, W - C)$ is the image of $1 \in \mathbb{Z}$ under the composition

$$\mathbb{Z} = H_m(\mathbb{S}^m) \to H_m(\mathbb{S}^m, \mathbb{S}^m - C) \to H_m(W, W - C)$$

where the first and second arrows are the homomorphisms induced by inclusion and the second one is an isomorphism, by excision.

Remark 3.8. The fundamental class $\mathcal{O}_C$ indeed does not depend on W: if U is any other neighborhood of C in $\mathbb{R}^m$, then the two inclusion maps $(W \cap U, (W \cap U) - C)$ into $(W, W - C)$ and $(U, U - C)$ send $\mathcal{O}_C$ to itself.

Let $P_n \subset \mathbb{R}^{d_n}$ be a polytope. Define $F := \times_n [P_n]$, where $[P_n]$ is the affine space generated by P_n, and let $F_0 := F - F$, where the symbol "$-$" denotes the subtraction in the vector space $\times_n \mathbb{R}^{d_n}$. The space F_0 is the unique linear subspace of $\times_n \mathbb{R}^{d_n}$ that is parallel to F and of the same dimension as F (see Rockafellar (2015)). Both F and F_0 are homeomorphic to a Euclidean space. Therefore the definition of fundamental class applies in the obvious way to compact subsets of these spaces. Let r be a retraction of F onto $P := \times_n P_n$. Then every Nash map extends to the map $f \circ (id_{\Pi_n A(P)} \times r)$ on $\Pi_n A(P) \times F$. If r' was another retraction of F onto P then, using a linear homotopy, r is homotopic to r'. This implies that the induced homotopy between $f \circ (id_{\Pi_n A(P)} \times r)$ and $f \circ (id_{\Pi_n A(P)} \times r')$ preserves the set of fixed points.

Let $\mathcal{E}(V)$ be set of equilibria of V. For each game V, $\mathcal{E}(V)$ is a compact semi-algebraic set and hence has finitely many connected components. This implies in particular that for each component C of $\mathcal{E}(V)$, there exists a neighborhood of C that is disjoint from $\mathcal{E}(V) - C$.

Definition 3.9. Fix a Nash-map f, a strategic-form game V, and a component C of $\mathcal{E}(V)$. Choose an open neighborhood W of C in F disjoint from $\mathcal{E}(V) - C$. Let $d : (W, W - C) \to (F_0, F_0 - 0)$ be the displacement map given by $d(p) = p - (f_V \circ r)(p)$. The index of C under f, denoted $\text{ind}(C, f)$, is the unique integer i for which $d_*(\mathcal{O}_C) = i \cdot \mathcal{O}_0$.

Remark 3.10. Note that it is implicit in the notation $\text{ind}(C, f)$ that C is a component of fixed points of f_V.

One example of a Nash-map is the map used by Nash in Nash (1951) to prove existence of equilibria. Another is the GPS-map in Gül, Pearce and Stacchetti (1993), whose definition we present in subsection 7.3 in the Appendix.

As seen from Definition 3.9, the Nash-map index of a component of equilibria C apparently depends on the specific Nash-map used to assign the index. We show below in Theorem 3.11 that the dependence is just apparent: a Nash-map used to assign indices to a certain equilibrium component *assigns the same index as* *any other Nash-map*. This result is implied by Theorem 2 in Demichelis and Germano (2000). We offer in the appendix a different proof, specific to the context of strategic-form games.

Theorem 3.11. *Let $f^1, f^2 : \Pi_n A(P) \times P \to P$ be two Nash-maps. Then for any equilibrium component C of the strategic-form game $V \in \Pi_n A(P)$, it follows that $ind(C, f^1) = ind(C, f^2)$.*

Proof. See Appendix, subsection 7.2. □

Proposition 3.12 shows that indices are not only invariant to the Nash-maps used to assign them but also invariant to different presentations of the same strategic-form game.

Proposition 3.12. *Let $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ and $V' = (N, (P'_n)_{n\in N}, (V'_n)_{n\in N})$ be two strategic-form games such that there exists an affine bijection $E_n : P_n \to P'_n$ with $V'_n \circ E = V_n$, for $E = \times_n E_n$. If $f_1 : \Pi_n A(P) \times P \to P$ and $f_2 : \Pi_n A(P') \times P' \to P'$ are two Nash-maps and C is an equilibrium component of V, then $E(C)$ is an equilibrium component of V', and $ind(C, f_1) = ind(E(C), f_2)$.*

Proof. See Appendix, subsection 7.2. □

The Nash-map index, analogously to Proposition 3.5 about the degree, offers a simple criterion to identify robustness. This is a known property of the index so we just provide a reference for a proof.

Proposition 3.13. *Let $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ be a strategic-form game and C a component of equilibrium of V. Let f be a Nash-map and assume $ind(f, C) \neq 0$. Let U be an open neighborhood of C disjoint from $\mathcal{E}(V) - C$. Then for any continuous map $f' : P \to P$ sufficiently close (uniformly) to f_V, there exists a fixed-point of f' in U.*

Proof. See O'Neill (1953). □

3.2.2. *Best-Reply Index Theory.* We present a third way of defining the index of equilbiria, now using the Best-reply correspondence of a strategic-form game. The *Best-reply Index* of a component is defined as follows.

Let C be a component of equilibria of the strategic-form game V. Let U be open in P and a neighborhood of C such that its closure $cl(U)$ satisfies $cl(U) \cap \mathcal{E}(V) = C$. Let W be an open neighborhood of the Graph(BR^V) such that $W \cap \{(\sigma, \sigma) \in P \times P | \sigma \in cl(U) - U\} = \emptyset$. By Corollary 2 in McLennan (1989), there exists $O \subset W$ a neighborhood of Graph(BR^V) such that any two continuous functions f_0 and f_1 from P to P whose graphs are in the neighborhood O are homotopic by a homotopy $H : [0, 1] \times P \to P$ with Graph$(H(t, \cdot)) \subset W$ for all $t \in [0, 1]$. By Corollary 1 of McLennan (1989), there exists a continuous map $f : P \to P$ with Graph$(f) \subset O$. We define the index of the best-reply of component C, denoted $\text{Ind}_{BR^V}(C)$, as the fixed point index of the continuous map $f|_U : U \to P$. The choice of the neighborhood W and the homotopy property of the index (see Dold (2012), Chapter VII, 5.15) imply that the index of the component is the same for any continuous map with graph in the neighborhood O.

The Best-reply index is defined through "perturbing" the best-reply to a nearby continuous map and using this map to assign indices in regular fashion. Therefore, an analogous version of Proposition 3.13 to the best-reply correspondence holds immediately, telling us that if the best-reply index of a component is nonzero, then that component is robust to small perturbations of the best-reply to any upper-hemicontinuous, contractible valued correspondences uniformly close to the best-reply. We denote the index of an equilibrium component C according to the best-reply BR^V of strategic-form game V as $\text{ind}_{BR^V}(C)$.

Theorem 3.14 shows that the best-reply index of a component of equilibria is also invariant under any presentation of a strategic-form game.

Theorem 3.14. *Let $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ and $V' = (N, (P'_n)_{n\in N}, (V'_n)_{n\in N})$ be two strategic-form games such that there exists an affine bijection $E_n : P_n \to P'_n$. Let $E = \times_n E_n$ and assume $V' \circ E = V$. Then if C is an equilibrium component of V then $E(C) = C'$ is an equilibrium component of V' and $\text{ind}_{BR^V}(C) = \text{ind}_{BR^{V'}}(C')$.*

Proof. The proof is a direct application of Theorem 7.13, in the Appendix. □

4. Equivalence between Index and Degree Theories in Strategic-form

The class of perturbations with respect to which Nash-map and Best-reply index theories capture robustness of equilbiria is much larger than the class with respect to which degree theory captures robustness: the maps with respect to which an equilibrium component might be robust might not even even be Nash-maps for any game, whereas the robustness captured by degree theory is centered in payoff perturbations of the game. However, the next result shows that index and degree numbers of a component of equilibria are the same – which implies that, in terms of capturing robustness to perturbations, both index and degree are equivalent.

Theorem 4.1. *Let $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ be a strategic-form game and C a component of equilibria of V. Then the Nash-map index, Best-reply index and degree of C are equal.*

Proof. See Appendix, subsection 7.3. □

5. Equivalence between Normal-form and Strategic-form Index and Degrees

Recall the definition of the map Π_n in section 2 - the map that identifies the equivalent mixed strategies of player n. The next result shows that no relevant information for robustness to payoff perturbation is lost when the equilibrium analysis is restricted to the strategic-form of a normal-form game. Corollary 5.2 solves a potential big issue with strategic-form index and degree theories: in principle, one does not know whether strategic-form degrees/indices and normal-form degrees/indices agree. Therefore, it could be the case that there exists C an equilibrium component of a normal-form game G with index zero – therefore a component that should be "discarded", since its robustness is not guaranteed – but $\times_n \Pi_n(C)$ is an equilibrium component of V – the strategic-form of G – with index nonzero, therefore a component that should be "selected" because of its robustness properties. This would create a fundamental ambiguity between what each theory of index and degree recommends in terms of equilibrium selection and would imply ultimately that the strategic-form of a game is not only an alternative depiction of the game but eliminates relevant information regarding robustness to payoff perturbation. Fortunately, Corollary 5.2 shows this is not possible.

Theorem 5.1. *Let $G = (N, (\Sigma_n)_{n\in N}, (G_n)_{n\in N})$ be a normal-form game and C an equilibrium component of G. If $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ is the strategic-form of G with payoff functions satisfying $V_n \circ (\times_n \Pi_n) = G_n, \forall n \in N$, then $(\times_n \Pi_n)(C) =: C'$ is a component of equilibria of V and $\text{ind}_{BR^G}(C) = \text{ind}_{BR^V}(C')$.*

Proof. The result follows from application of Theorem 7.13 and Corollary 7.14, in the Appendix. ☐

Corollary 5.2. *Let G, V, C and C' be as in Theorem 5.1. Then $deg_V(C') = deg_G(C) = ind_{BR^V}(C') = ind_{BR^G}(C)$.*

Proof. See Appendix, subsection 7.4. ☐

5.1. **How to Compute Degrees.** The result of Corollary 5.2 on the equivalence of strategic-form index/degree theories and normal-form index/degree theories shows us that any of these theories can be used to calculate the degree or index of an equilibrium component. However, the calculation of local homology groups and homology homomorphisms required for the calculation of the degree of a component of equilibria is in most cases not a straightforward one. We present in this subsection a simpler way to calculate degree.

Let $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ be a strategic-form game and let C be a component of equilibria of V. We call a polytope $P_n^s \subset \mathbb{R}^{d_n}$ standard if P_n^s is contained in the affine space generated by the unit simplex of $\mathbb{R}^{d_n}$ and if P_n^s has dimension $d_n - 1$. Notice that if P_n is not standard than there exists a polytope $P_n^s \subset \mathbb{R}^{d_n}$ for some $d_n \in \mathbb{N}$ such that P_n^s is standard and there exists an affine bijection $e_n : P_n^s \to P_n$. Now, defining $V_n^s := V_n \circ (\times_n e_n), \forall n \in N$, it follows from Proposition 3.4 that $\deg_V(C) = \deg_{V^s}(C^s)$, where $C^s := (\times_n e_n)(C)$. Thus the problem of calculating the degree of the equilibrium component C of the arbitrary strategic-form game V can be reduced to the problem of calculating the degree of C^s of a *standard* strategic-form game obtained from V - a strategic-form game where the strategy polytopes of each player are standard.

Assume now $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ is such that $\forall n \in N, P_n$ is standard. The payoff functions V_n for each player $n \in N$ are now *multilinear* and can be identified with a vector in $\mathbb{R}^{NM}$ where $M = d_1 ... d_N$. We reduce now the problem of calculation of the degree of a component even further.

Let $V \bigoplus g$ be the standard strategic-form game whose payoff functions are given by $V_n(p_n, p_{-n}) + p_n \cdot g_n$, $g_n \in \mathbb{R}^{d_n}$ and $\mathcal{E}_V := \{(g, p) \in \Pi_n \mathbb{R}^{d_i} \times P |\ p \text{ is an equilibrium of } V \bigoplus g\}$. Because V is standard, the payoffs $V_n(p_n, p_{-n})$ can be written as $p_n \cdot V^n(p_{-n})$, where $V^n(p_{-n}) = \nabla_{p_n} V_n(p_n, p_{-n})$. Now notice that there is a Structure Theorem for $\mathcal{E}_V$ that is exactly analogous to the one obtianed in Theorem 3.1: let $\theta_V : \mathcal{E}_V \to \Pi_n \mathbb{R}^{d_n}$ be defined by $(\theta_V)_n(g, p) = p_n + V^n(p_{-n}) + g_n \in \mathbb{R}^{d_n}$. It follows that θ_V is continuous and it can be verified that it has a continuous inverse: $(\theta_V)^{-1}(z) = (h(z), r(z))$, where $h_n(z) = z_n - p_n - V^n(p_{-n})$ and $p_j = r_j(z_j)$, $\forall j \in N$, where r_j is the nearest-point retraction to the standard polytope P_j. Also, $\text{proj}_g \circ \theta_V^{-1}$ is homotopic to the identity function in $\Pi_n \mathbb{R}^{d_n}$, where $\text{proj}_g : \mathcal{E}_V \to \Pi_n \mathbb{R}^{d_n}$ is the projection over the first coordinate.

It follows from the proof of Lemma 7.10 that the degree of C can be calculated from the function $\text{proj}_g \circ \theta_V^{-1}$ as follows: let $\theta_V(0, C) = K$. If U is an open neighborhood of K such that its closure $\text{cl}(U)$ contains no other z such that $h(z) = 0$ besides those in K, then $\deg_V(C)$ equals the local degree of $\text{proj}_g \circ \theta_V^{-1}|_U$ over 0.

The problem of calculating the degree of C is therefore reduced to calculating the degree over 0 with respect to the map $\text{proj}_g \circ \theta_V^{-1}|_U$, which is a function defined on the Euclidean space $\Pi_n \mathbb{R}^{d_n}$. We will show now how to calculate the local degree of $\text{proj}_g \circ \theta_V^{-1}|_U$ over 0 by approximating the game V with "generic" games.

We first define precisely what genericity means in this context. Let $V_n(P_n)$ be the vertex set of polytope P_n and let $T_n \subset V_n(P_n)$ be the subset of vertices that generates a face $[T_n]$ of the polytope P_n. The restriction of $\text{proj}_g \circ \theta_V^{-1}$ to the set of $z \in \Pi_n \mathbb{R}^{d_n}$ such that for each n, $r_n(z_n)$ is in the relative interior of the face

$[T_n]$, is a polynomial map of degree $|N|-1$. Let g be *generic* if g is a regular value of each polynomial map obtained through this restriction. The set of regular values is then open and dense in $\Pi_n \mathbb{R}^{d_n}$, by Sard's Theorem. We say that a game $V' = V \bigoplus g$ is generic if g is chosen generically. As a consequence of the inverse function theorem and our definition of genericity, it follows that a generic strategic-form game has finitely many equilibria.

For a generic game V, let p be an equilibrium of this game and let $\theta_V(0,p) = z$. Then the local degree of $\text{proj}_g \circ \theta_V^{-1}|_U$ over 0 is given by $\text{sign}(\det[D(\text{proj}_g \circ \theta_V^{-1})(z)])$. The computation of the degree of p can therefore be done explicitly through the computation of the determinant of the jacobian $D(\text{proj}_g \circ \theta_V^{-1})(z)$ and then checking its sign. This jacobian is a square matrix of dimension $d_1 + ... + d_n$.

Let now V be nongeneric, standard and fixed. Assume C is a component of equilibrium of V and U an open neighborhood of $K = \theta(0, C)$, defined as before. From Proposition 5.12, Chapter IV, in Dold (2012), it follows the the $\deg_V(C)$ is locally constant in V. It implies that for a generic perturbation $g \in \Pi_n \mathbb{R}^{d_n}$ sufficiently close to 0, the game $V \bigoplus g$ has finitely many equilibria and the local degree of $\text{proj} \circ \theta_V^{-1}|_U$ over g equals the local degree of $\text{proj} \circ \theta_V^{-1}|_U$ over 0. The additivity property of the degree (see Dold (2012), Proposition 5.8, Chapter IV) now implies that the local degree of $\text{proj} \circ \theta_V^{-1}|_U$ over g is the summation:

$$\sum_{z \in U : z \in h^{-1}(g)} \text{sign}(\det[D(\text{proj}_g \circ \theta_V^{-1})(z)]).$$

This shows therefore that:

$$\deg_V(C) = \sum_{z \in U : z \in h^{-1}(g)} \text{sign}(\det[D(\text{proj}_g \circ \theta_V^{-1})(z)]).$$

The formula above shows how the calculation of the degree of a component of equilibria depends on the dimension of the strategy polytopes: the dimension of the jacobian matrix $[D(\text{proj}_g \circ \theta_V^{-1})(z)]$ at z is $d_1 + ... + d_n$. Typically the number of pure strategies of a player in the normal-form representation of an extensive-form game grows exponentially with the size of the tree. If the formula above is used for computation of the degree in the normal-form of an extensive-form game, the dimension of the jacobian matrix is therefore typically exponential in the number of terminal nodes of the tree, which is an undesirable property if one is interested in extensive-form games with a particularly large number of terminal nodes. In subsection 6.2, we present a strategic-form representation of an extensive-form game called the *strategic-form of the extensive-form*. The dimension of the strategy polytopes in this representation is linear in the size of the tree, which makes the computation of the degree much less expensive than in normal-form.

6. Extensive-form games and Degree Theory

The strategic-form of a game is derived through a process of identification of equivalent mixed strategies, which by definition depends on payoffs of the game. In the case of extensive-form games, it is possible to perform an analogous identification which does not rely on payoffs, but exclusively on the game form. We obtain then the *strategic-form of an extensive-form game*. This concept is presented in the next subsection together with necessary notation which will be used further ahead in the paper.

6.1. **Preliminaries on Extensive Form Games.** We consider the space $\mathcal{G}$ of extensive-form games obtained by assigning payoffs to the final nodes of a finite game tree $\Gamma := (T, \prec, U, N, P_*)$ with perfect recall. The set T is the set of nodes and $\prec$ is the irreflexive binary relation of precedence in the tree $(T, \prec)$; that is, the relation $\prec$ is acyclic and totally orders the predecessors $\{t^{'}|t^{'} \prec t\}$ of t. The subset of terminal nodes – those with no successors – is $Z \subset T$, U is a partition of $T \setminus Z$ into information sets of players and nature. The set N denotes the set of players. The set $U_n \subset U$ is the collection of information sets for player $n \in N$ and $A_n(u)$ is n's set of actions available at his information set $u \in U_n$. Let $A_n = \cup_{n \in N} A_n(u)$ be the entire set of n's actions. Write $u \prec z$ if $t \prec z \in Z$, for some node $t \in u$, and write $(u, i) \prec z$ if there exists $t \prec t^{'} \preceq z$ for some node $t^{'}$ that follows $t \in u$ and action $i \in A_n(u)$. Similarly $i \prec i^{'}$ if $i, i^{'}$ are actions at $u, u^{'}$ with $(u, i) \prec u^{'}$. Perfect recall implies that each $(U_n, \prec)$ is a tree. Player n set of pure strategies is $S_n := \{s : U_n \to A_n | s(u) \in A_n(u)\}$ and his simplex of mixed strategies is $\Sigma_n := \Delta(S_n)$. Kuhn (1950) shows that in a game tree with perfect recall each player n can implement a mixture of pure strategies by a payoff-equivalent behavior strategy $b_n = (b_n(u))_{u \in U_n}$ in which each $b_n(u) \in \Delta(A_n(u))$ is a mixture of actions in $A_n(u)$; i.e., $b_n(i|u)$ is the conditional probability at u that n chooses i.

For the fixed tree Γ, the space of games will be denoted $\mathcal{G} := \mathbb{R}^{N|Z|}$, where a game $G \in \mathcal{G}$ assigns payoff $G_n(z)$ to player n at final node z. The space of outcomes is $\Omega = \Delta(Z)$, where an outcome $P \in \Omega$ assigns probability $P(z)$ to z. The probability $P_*(z) > 0$ is the probability that Nature's actions do not exclude the final node z. The probability $P_*(z) > 0$ is formally defined as follows: consider Nature as a player that plays a fixed behavior strategy. Then fix any mixed strategy $\sigma_* \in \Delta(S_*)$ that is equivalent to this behavior strategy, where S_* are the "pure strategies" of Nature. Let $S_*(z) = \{s \in S_* | (u, i) \prec z \Rightarrow s(u) = i\}$. Then $P_*(z) := \sum_{s_* \in S_*(z)} \sigma_*(s_*)$.

From a mixed strategy profile σ one can derive the corresponding *enabling* profile $p \in [0,1]^{L_n}$ as follows. Let L_n is the set of players n's last actions: $i \in L_n \subset A_n$ if there exists $z \in Z$ such that i is the $\prec$-maximal element in $A_n(z) := \{i^{'} \in A_n | i^{'} \prec z\}$. That is $i = l_n(z) = \text{argmax} A_n(z)$. If $L_n = \emptyset$, then n is a dummy player, so p_n can be omitted from the profile. For each $i \in L_n$, $p_n(i)$ is the probability under σ_n that n's selected pure strategy does not exclude i or any of n's actions preceding i. One defines $p_n(i)$ as follows. The subset of n's pure strategy that does not exclude z is $S_n(z) := \{s \in S_n | (u, i) \prec z \text{ implies } s(u) = i\}$. If n uses the mixed strategy σ_n, then the probability that n does not exclude z is $P_n(z) = \sum_{s \in S_n(z)} \sigma_n(s)$ or $P_n(z) = 1$, if $A_n(z) = \emptyset$. Define $Z_n(i) := l_n^{-1}(i) = \{z | l_n(z) = i\}$ and $s_n(i) := S_n(z)$, for all $z \in Z_n(i)$. Then $p_n(i) := Prob_\sigma(s_n(i)) = P_n(z)$, for each $z \in Z_n(i)$. The set of enabling profiles is $C = \times_n C_n$, where for each non-dummy player n, his set of enabling strategies is

$$C_n = \{p_n \in [0,1]^{L_n} | (\exists \sigma_n \in \Sigma_n)(\forall i \in L_n), p_n(i) = \sum_{s \in s_n(i)} \sigma_n(s)\}.$$

Let $\Sigma_n \subset \mathbb{R}^{m_n}$ be the unit simplex. Observe that C_n is a polytope: indeed, let $e_i = (0, ..., 1, ..., 0) \in \mathbb{R}^{L_n}$ and $v_j = (0, .., 1, ..., 0) \in \mathbb{R}^{m_n}$ where the number 1 is in the i-th position and j-th position, respectively. If $p_n \in C_n$ then there exists $\sigma_n \in \Sigma_n$ such that $p_n(i) = \sum_{s \in s_n(i)} \sigma_n(s)$ for all $i = 1, ..., L_n$. Therefore the pair (p_n, σ_n) is a solution to the following system of linear inequalities in $(p_n, \sigma_n) \in \mathbb{R}^{L_n} \times \mathbb{R}^{m_n}$:

$$1 \geq e_i \cdot p_n \geq 0, \forall i \in \{1, ..., L_n\}. \tag{1}$$

(2) $$1 \geq v_{s_n} \cdot \sigma_n \geq 0, \forall s_n \in S_n.$$

(3) $$\sum_{s_n \in S_n} v_{s_n} \cdot \sigma_n = 1.$$

(4) $$e_i \cdot p_n - \sum_{s_n \in s_n(i)} \sigma_n(s_n) = 0, \forall i.$$

Constraints 1 - 4 show that the set of solutions S to the linear system above is bounded, which implies S is a polytope. Let $\mathrm{proj}_{\mathbb{R}^{L_n}} : \mathbb{R}^{L_n} \times \mathbb{R}^{m_n} \to \mathbb{R}^{L_n}$ be defined by $\mathrm{proj}_{\mathbb{R}^{L_n}}(p_n, \sigma_n) = p_n$. If $S \subset \mathbb{R}^{L_n} \times \mathbb{R}^{m_n}$ is the set of solutions to the system, then $\mathrm{proj}_{\mathbb{R}^{L_n}}(S) = C_n$. Because S is a polytope and $\mathrm{proj}_{\mathbb{R}^{L_n}}$ is affine, it follows that C_n is a polytope.

Notice that the definition of enabling strategies implies that p_n is not necessarily a probability: given $\sigma_n \in \Sigma_n$, the p_n associated to it is such that $p_n(i)$ is the sum of σ_n-probabilities over the pure strategies that choose a sequence of moves leading to last action i. But a single pure strategy s_n might lead to two different last actions i and i' simultaneously, so when we sum the $p_n(i)$'s over i, the $\sigma_n(s_n)$ is counted at least twice, which implies that this sum might be strictly larger than 1. This is what happens in the strategy polytope of player II in Example 2.2.

The next result establishes that the enabling strategy set is also the result of a specific identification of equivalent mixed-strategies.

Proposition 6.1. *The enabling strategy set C_n is a quotient-space of the set of mixed-strategies Σ_n.*

Proof. Let $q_n : \Sigma_n \to C_n$ defined by $q_n(\sigma_n) = p_n$, where $p_n(i) = \sum_{s \in s_n(i)} \sigma_n(s), \forall i \in L_n$. Let Σ_n and C_n be endowed with their relative topologies. The map q_n is clearly continuous and is surjective by definition. Therefore, by the Closed Map Lemma[9], since Σ_n is compact and C_n is in particular Hausdorff, it implies q_n is a quotient map. □

Corollary 6.2. *Let $\mathring{C}_n := \{p_n \in [0,1]^{L_n} | (\exists \sigma_n \in \mathrm{int}(\Sigma_n))$ s.t. $p_n(i) = \sum_{s \in s_n(i)} \sigma_n(s)\}$. Then $\mathring{C}_n = \mathrm{int}(C_n)$.*

Proof. It it clear that $\mathring{C}_n \subset \mathrm{int}(C_n)$. So we show the converse. First, $\mathrm{int}(C_n)$ is open in C_n so $q_n^{-1}(\mathrm{int}(C_n))$ is open in Σ_n, by continuity of q_n. It implies $q_n^{-1}(\mathrm{int}(C_n)) \subset \mathrm{int}(\Sigma_n)$. Therefore if $p_n \in \mathrm{int}(C_n)$, then $q_n^{-1}(p_n) \subset \mathrm{int}(\Sigma_n)$. Hence there exists $\sigma_n \in \mathrm{int}(\Sigma_n)$ such that $q_n(\sigma_n) = p_n \iff p_n(i) = \Sigma_{s \in s_n(i)} \sigma_n(s)$, for all $i \in L_n$. So $p_n \in \mathring{C}_n$. □

6.2. **The Strategic-form of an Extensive-form Game.** The objective of this section is to show that an extensive-form game $G \in \mathcal{G}$ obtained from a game-tree Γ admits a unique strategic-form representation in which the strategy sets of the players are the polytopes of enabling strategies. This is Theorem 6.8.

We fix in this subsection a game tree Γ that has no chance moves. For the case when it does, the proof of Theorem 6.8 will be exactly the same: include Nature as a player – without payoffs – and proceed to derive payoffs for the other players in the same manner as indicated in the proof of Theorem 6.8.

[9]This is Lemma 4.50 in Lee (2010). It states among other things that a continuous, surjective map from a compact space to a Hausdorff space is a quotient map.

Definition 6.3. A profile of behavior strategies $b = (b_n)_{n\in N}$ *induces an enabling profile* $p = (p_n)_{n\in N}$ if any mixed strategy profile $\sigma = (\sigma_n)_{n\in N}$ that is equivalent to b [10] satisfies $p_n(i) = \sum_{s\in s_n(i)} \sigma_n(s), \forall i \in L_n$.

The next proposition establishes the relation between enabling and behavior strategies.

Proposition 6.4. *The following hold:*

(1) Given a profile of behavior strategies $b = (b_n)_{n\in N}$, there exists a unique profile of enabling strategies induced by b.

(2) Let $(p_n)_{n\in N}$ be a profile of enabling strategies with $p_n \in \mathring{C}_n$. Then there exists a unique profile of behavior strategies that induces $(p_n)_{n\in N}$.

Proof. We prove (1). Given a profile of behavior strategies b, define $p_n(i) := \Pi_{(u^{'},i^{'})\preceq(u,i)} b_n(i^{'}|u^{'})$ for each $i \in L_n$. Let σ be a profile that is equivalent to b. Then equivalence implies that, for each $n \in N, i \in L_n$, $\sum_{s\in s_n(i)} \sigma_n(s) = \Pi_{(u^{'},i^{'})\preceq(u,i)} b_n(i^{'}|u^{'})$. Therefore $p_n \in C_n$ and b induces p. Uniqueness follows immediately.

We prove (2). Let $(p_n)_{n\in N}$ be an enabling profile with $p_n \in \mathring{C}_n$. Then there exists $\sigma_n \in \text{int}\Sigma_n$ satisfying $p_n(i) = \sum_{s\in s_n(i)} \sigma_n(s)$. Define $\beta_n(u, i) := p_n(i)$, where $i \in A_n(u)$ is a last action of player n. Then, if $(u^{'}, i^{'})$ is an immediate predecessor of u among $n's$ information set, define $\beta_n(u^{'}, i^{'}) := \sum_{i\in A_n(u)} \beta_n(u, i)$. Proceeding in this manner, we define $\beta_n(u, i)$ for every pair $(u, i) \in U_n \times A_n(u)$. Notice that because of the assumption $p_n \in \mathring{C}_n$, it follows that $\beta_n(u, i) > 0$ for all $u \in U_n$ and $i \in A_n(u)$. Therefore we can define the behavior strategy $b_n(i|u) := \frac{\beta_n(u,i)}{\beta_n(u^{'},i^{'})}$, where $(u^{'}, i^{'}) \prec u$ and $u^{'}$ is an immediate predecessor of u. Now, for any $\tilde{\sigma}_n$ equivalent to b_n, it must be that $\sum_{s\in s_n(i)} \sigma_n(s) = \sum_{s\in s_n(i)} \tilde{\sigma}_n(s), \forall i \in L_n$, otherwise we can construct $\sigma^{'}_{-n}$ such that $(\sigma_n, \sigma^{'}_{-n})$ and $(\tilde{\sigma}_n, \sigma^{'}_{-n})$ do not induce the same distributions over terminal nodes. This implies that $(b_n)_{n\in N}$ induces $(p_n)_{n\in N}$. Now, for uniqueness, suppose b' is a profile of behavior strategies inducing $(p_n)_{n\in N}$. By the proof of (1), it follows that $p_n(i) = \Pi_{(u^{'},i^{'})\preceq(u,i)} b'_n(i^{'}|u^{'}), \forall i \in L_n$. For each $u \in U_n$ and $i \in L_n$ such that $i \in A_n(u)$ set $\beta'_n(u, i) = p_n(i)$. Proceeding in the same fashion as we did for β_n, the numbers $\beta'_n(u', i') > 0$ for each $u' \in U_n$ and $i \in A_n(u')$ are uniquely determined. This implies therefore that for each $u \in U$ and $i \in A_n(u)$, $b'_n(i|u) = b_n(i|u)$, which shows uniqueness. □

Remark 6.5. Notice that for each $p_n \in C_n$, there exists a behavior strategy profile $(b_n)_{n\in N}$ inducing $(p_n)_{n\in N}$, but this behavior strategy need not be unique. This happens when certain last actions have probability 0 for a certain player. Still, whenever we have $p_n(i) > 0$, it is possible to proceed as in the proof of (2) Proposition 6.4 and derive $b_n(i'|u')$ for each $(u', i') \preceq (u, i), i \in A_n(u)$. For the remaining pairs (u, i), the probabilities $b_n(i|u)$ are undetermined.

Definition 6.6. An enabling profile $p = (p_n)_{n\in N}$ induces a distribution over terminal nodes $P \in \Delta(Z)$ if for any profile of behavior strategies $(b_n)_{n\in N}$ inducing p, it implies that $(b_n)_{n\in N}$ induces P.

Corollary 6.7. *Let $(p_n)_{n\in N}$ be a profile of enabling strategies with $p_n \in C_n$. Then there exists a unique distribution $P \in \Delta(Z)$ induced by this profile of enabling strategies. Conversely, given $P \in \text{int}\Delta(Z)$, there exists a unique profile $(p_n)_{n\in N}$, with $p_n \in \mathring{C}_n$, that induces P.*

[10] A mixed strategy profile σ is equivalent to a behavior profile b if for each n, σ_n is equivalent to b_n. The mixed strategy σ_n is equivalent to b_n if for any mixed/behavior profile σ_{-n} the distribution over terminal nodes induced by (σ_n, σ_{-n}) and (b_n, σ_{-n}) is the same (see Maschler, Solan and Zamir (2013), p. 223).

Proof. The first part of the statement is straightforward by an application of (1) of Proposition 6.4. We prove the second part. Given $P \in \text{int}\Delta(Z)$ we show there exists a unique behavior strategy profile $(b_n)_{n\in N}$ that induces P. Let i be an action of player n at an information set $u \in U_n$ and define

$$b_n(i|u) := \frac{\sum_{z:(u,i)\prec z, u\in U_n} P(z)}{\sum_{z:u\prec z, u\in U_n} P(z)}.$$

This defines the unique behavior strategy b_n and the profile $(b_n)_{n\in N}$ induces P. Also, by the proof of (1) in Proposition 6.4, $(b_n)_{n\in N}$ induces a unique enabling profile $(p_n)_{n\in N}$ with $p_n(i) = \Pi_{(u',i')\preceq(u,i)} b_n(i'|u') > 0$ for each $i \in L_n, n \in N$. ☐

Theorem 6.8. *Let C_n denote the enabling strategy set of player n derived from game-tree Γ. Then there exists a linear subspace $A^\circ(\times_n C_n)$ of the multiaffine functions over $\times_n C_n$ such that:*

1. *For any $(V_n)_{n\in N} \in \Pi_{n\in N} A^\circ(\times_n C_n)$ there exists a unique $G \in \mathcal{G}$, such that for any profile of enabling strategies $(p_n)_{n\in N}$ and induced distribution $P \in \Delta(Z)$, $V_n(p_n, p_{-n}) = \sum_{z\in Z} G_n(z)P(z)$.*
2. *Conversely, for each $G \in \mathcal{G}$ there exists a unique multiaffine function $V_n \in A^\circ(\times_n C_n)$ for each $n \in N$ such that for any profile of enabling strategies $(p_n)_{n\in N}$ and induced distribution $P \in \Delta(Z)$, $V_n(p_n, p_{-n}) = \sum_{z\in Z} G_n(z)P(z)$.*

Proof. We first construct the linear subspace $A^0(\times_{n\in N} C_n)$ of the statement. For each $z \in Z$, there exists an unique path in Γ from the root to z. Therefore, for each $z \in Z$, there exists a unique pair of set $N^*(z) \subset N$ and vector $(i_n)_{n\in N^*(z)}$ of last actions such that $l_n(z) = i_n \in L_n$, for $n \in N^*(z)$ and $l_n(z) = \emptyset$, for $n \in N \setminus N^*(z)$. We call this vector *the unique vector of last actions associated to z.* Let $W := \{(i_n)_{n\in N^*} | \exists z \in Z \text{ s.t. } N^* = N^*(z)\}$. Define the multiaffine function V_n over $\times_n C_n$ by:

$$V_n(p_1, ..., p_N) := \sum_{(i^*_{j_1},...,i^*_{j_n})\in W} V_n(i^*_{j_1}, ..., i^*_{j_n}) p_{j_1}(i^*_{j_1})...p_{j_n}(i^*_{j_n})$$

Notice that the set of affine functions satisfying the formula above forms a linear subspace of the space of multiaffine functions over $\times_n C_n$ (under pointwise addition and scalar multiplication). Call this subspace $A^\circ(\times_n C_n)$.

We prove (1). Let $V_n \in A^\circ(\times_n C_n)$. We now show that this function defines unique payoffs over terminal nodes of Γ for player n. For $z \in Z$, consider the unique vector of last actions $(i_n)_{n\in N^*(z)} \in W$ associated to z. Define, for each $h \in N$, $G_h(z) := V_h(i^*_{j_1}, ..., i^*_{j_n})$. Let $(p_n)_{n\in N}$ be a profile of enabling strategies and P the induced distribution over Z. For $z \in Z$, if $(i^*_{j_1}, ..., i^*_{j_n}) \in W$ is associated to z, then it implies that:

$$P(z) = P_{j_1}(z)...P_{j_n}(z) = p_{j_1}(i^*_{j_1})...p_{j_n}(i^*_{j_n}),$$

where P_{j_i} are the probability distributions over terminal nodes defined at the beginning of section 6.

Then $G_n(z)P(z) = V_n(i^*_{j_1}, ..., i^*_{j_n}) p_{j_1}(i^*_{j_1})...p_{j_n}(i^*_{j_n})$. This implies that

$$\sum_{z\in Z} G_n(z)P(z) = \sum_{(i^*_{j_1},...,i^*_{j_n})\in W} V_n(i^*_{j_1}, ..., i^*_{j_n}) p_{j_1}(i^*_{j_1})...p_{j_n}(i^*_{j_n}) \tag{5}$$

Now we show (2). For each $z \in Z$, consider the unique $(i^*_{j_1}, ..., i^*_{j_n}) \in W$ associated to z. Define $V_n(i^*_{j_1}, ..., i^*_{j_n}) := G_n(z)$. Define the multiaffine function for player $n \in N$ by:

$$V_n(p_1, ..., p_N) := \sum_{(i^*_{j_1}, ..., i^*_{j_n}) \in W} V_n(i^*_{j_1}, ..., i^*_{j_n}) p_{j_1}(i^*_{j_1}) ... p_{j_n}(i^*_{j_n}).$$

Then V_n belongs to $A^\circ(\times_n C_n)$ and expected payoffs agree. □

Remark 6.9. For each player n, let $(G_n(z))_{z \in Z}$ be the vector payoffs of player n associated to terminal nodes of the tree. We will use in the last subsection a similar notation to Govindan and Wilson (2002) and write $V_n(p_n, p_{-n}) = p_n \cdot \nu_n(p_{-n}) + \nu_n(\emptyset)$ where $\nu_n(p_{-n}) := (\nu_n(i, p_{-n}))_{i \in L_n}$. Formally, given a vector of enabling strategies $(p_n)_{n \in N} \in \Pi_n C_n$ and induced distribution $P \in \Delta(Z)$, we have

$$V_n(p_n, p_{-n}) = \sum_{z \in Z} G_n(z) P(z) =$$

$$\sum_{i \in L_n} p_n(i) \sum_{z \in Z_n(i)} G_n(z) P^n(z) + \sum_{z | A_n(z) = \emptyset} G_n(z) P^n(z) = p_n \cdot \nu_n(p_{-n}) + \nu_n(\emptyset),$$

where $P^n(z) = \Pi_{j \neq n} P_j(z)$. Hence $p_n \cdot \nu_n(p_{-n})$ corresponds to the part of the multiaffine function V_n that depends on the last actions of player n, whereas $\nu_n(\emptyset)$ depends exclusively on p_{-n}.

An immediate consequence of the proof of the Theorem 6.8 above is the following.

Corollary 6.10. *Let $C = \times_n C_n$ and $R : \Pi_{n \in N} A^\circ(C) \to \mathbb{R}^{N|Z|}$ defined by $R(V) = G$, where $R := \times_n R_n$ and $R_n(V_n) = G_n \in \mathbb{R}^{|Z|}$, in which G_n is the unique vector of terminal payoffs obtained in (1) of Theorem 6.8. Then R is a linear isomorphism.*

Definition 6.11. The *strategic-form of the extensive-form game $G \in \mathcal{G}$* is the strategic-form game $(N, (C_n)_{n \in N}, (V_n)_{n \in N})$ where the set of strategies of player n is the enabling strategy polytope C_n and the payoff function $V_n \in A^\circ(C)$ for each player n satisfies $R(V) = G$.

The representation of an extensive-form game as a strategic-form game with enabling strategies has two immediate advantages: first, it preserves the convexity of the space of mixed-strategies and the linearity of a player's payoff function in its own strategies. Second, by Corollary 6.10, the dimension of the space of payoffs of the strategic-form of a given extensive-form game is the same as the dimension of the space of terminal payoffs of the game-tree. It also follows from the construction of the enabling strategy set that given a terminal node of the game-tree, a player has at most one last action leading to that terminal node, which implies the dimension of the enabling strategy set of a player is at most the number of terminal nodes in the tree.

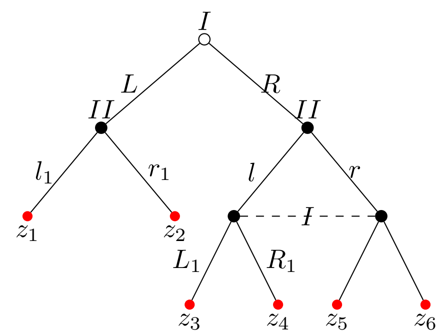
The strategic-form polytopes defined in section 2 were defined through a different process of identification of mixed-strategies that relied in a quotient map depending of payoffs of the normal-form. In the strategic-form of an extensive-form, the process of identification depends on the game-form. However, the strategic-form polytopes derived from the normal-form and the enabling polytope are isomorphic, if the payoffs in the game-tree are generic. More generally, the strategic-form polytopes derived from the normal-form perform "more identifications" than those performed in the enabling strategy polytopes, precisely because the strategic-form polytopes take payoffs into account, whereas enabling strategies are derived from the game-form only.

Theorem 6.12. *Let G be an extensive-form game and $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ be the strategic-form derived from the normal-form $\tilde{G}$ of G. If $V' = (N, (C_n)_{n\in N}, (V'_n)_{n\in N})$ is the strategic-form of the extensive-form game G then:*

1. *There exists a affine surjection $\bar{q}_n : C_n \to P_n$ and $V_n \circ (\times_n \bar{q}_n) = V'_n$, $\forall n \in N$.*
2. *If C' is an equilibrium component of V', then $\times_n q_n(C') := C$ is an equilibrium component of V and $\deg_V(C) = \deg_{V'}(C')$.*

Proof. See Appendix, subsection 7.5 □

Figure 3. Game-Tree of Example 2.2



Example 6.13. We show how to define payoff functions for player I in the strategic-form of the extensive-form game depicted in Figure 3, which is the same game-tree as that of Example 2.2. The payoff functions for player II can be defined using a similar procedure. Since the identification of mixed-strategies resulting in the enabling strategy set does not rely on payoffs, but only in the game-tree, we substitute the specific terminal payoffs of Example 2.2, by arbitrary terminal payoffs. Let $(G_I(z_i))_{i=1,..,6}$ be a vector of payoffs of player 1 over terminal nodes defined for the game-tree in Figure 3. Then define:

1. $V_I(L, \ell_1) := G_I(z_1)$ and $V_I(L, r_1) := G_I(z_2)$;
2. $V_I(\ell, L_1) := G_I(z_3)$ and $V_I(\ell, R_1) := G_I(z_4)$;
3. $V_I(r, L_1) := G_I(z_5)$ and $V_I(r, R_1) := G_I(z_6)$;

Thus,

$$V_1(p) = V_1(L, l_1)p_1(L)p_2(l_1) + V_1(L, r_1)p_1(L)p_2(r_1) + V_1(l, L_1)p_1(L_1)p_2(l) +$$
$$V_1(l, R_1)p_1(R_1)p_2(l) + V_1(r, L_1)p_1(L_1)p_2(r) + V_1(r, R_1)p_1(R_1)p_2(r),$$

where $p_1 \in C_1$ and $p_2 \in C_2$. In this game-tree, player I has 4 pure strategies, so his mixed strategy space is a 3-simplex. But the enabling strategy space C_1 is a 2-simplex. Now, player II also has 4 pure strategies but the enabling strategy set C_2 is 2-dimensional (it is actually the "square" of Example 2.2). The strategy space $\Sigma_1 \times \Sigma_2$ in normal-form has dimension 9 whereas $C_1 \times C_2$ has dimension 4.

6.2.1. *A Comment on Computational Advantages of the Strategic-form of the Extensive-form.* In Von Stengel (1996), the polytope of *realization-plans* is constructed as an alternative to behavior strategies. The reduced form of the realization-plan (see p.236 Σ 2 of Von Stengel (1996)) presented in that paper *is* the enabling strategy polytope. Following the same procedure outlined in that paper, the enabling strategy set can be obtained directly from the game-tree without having to compute the normal-form.

Recall from subsection 5.1, that the degree of a component of equilibrium can be calculated using the function $\text{proj}_g \circ \theta_V^{-1} : \Pi_n \mathbb{R}^{d_n} \to \Pi_n \mathbb{R}^{d_n}$, where $d_n - 1$ is the dimension of the standard polytope P_n^s. Starting from the strategic-form of the extensive-form $(N, (C_n)_{n\in N}, (V_n)_{n\in N})$ and defining a standard polytope P_n^s which is affinely bijective with C_n by a bijection $e_n : P_n^s \to C_n$, it implies then $d_n \leq L_n$. Therefore, the jacobian of $\text{proj}_g \circ \theta_V^{-1}(z)$ at a regular point z has dimension $\sum_{i\in N} d_i \leq N \max_{i\in N} |L_i| \leq N|Z|$ - that is, linear in $|Z|$.

The property of linearity with respect to terminal nodes can also be used in algorithms dedicated to compute equilibria in order to implement them faster. Because of Theorem 3.1, the results in Govindan and Wilson (2003) apply almost verbatim to the environment of strategic-form games, which implies that the Global Newton Method outlined in that paper also applies to strategic-form representation without any changes. However, the fact that the dimension of enabling polytopes is linear in the size of the tree will imply faster implementation of the Global Newton Method.

The GW-Structure Theorem. Govindan and Wilson (2002) present a structure theorem for strategic-form games, where the payoff space of the Nash-graph is the space of terminal payoffs of a fixed - but arbitrary - game-tree. The formulation of such a structure theorem over terminal payoffs of the tree has however a fundamental limitation: the strategy set of each player has to be perturbed. The perturbation is necessary for the statement of the structure theorem to be true for *any* game tree - as there are counterexamples otherwise (see Govindan and Wilson (2002)). Nevertheless, such a formulation of the structure theorem is the natural one for extensive-form games, because it involves terminal payoffs of the tree directly. Therefore, a natural question is how are the degrees assigned according to the structure theorem for extensive-form games of Govindan and Wilson related to the degrees assigned by the structure theorem for strategic-form games and normal-form games: do they capture essentially the same robustness properties to payoff perturbations? This is the main question we answer with Theorem 6.19 in this subsection.

Fix a game tree Γ with terminal payoffs $G \in \mathcal{G} := \mathbb{R}^{N|Z|}$.

Let $\tilde{G} := (N, (\Sigma_n)_{n\in N}, (\tilde{G}_n)_{n\in N})$ be the normal-form representation of the extensive-form game G. For $\epsilon > 0$, consider $P_n \subset \text{int}(\Sigma_n)$ a polytope in Σ_n such that the Hausdorff-distance $d(P_n, \Sigma_n) \leq \epsilon$. Let $P := \times_n P_n$. Then $G|_P = (N, (P_n)_{n\in N}, (G_n|_P)_{n\in N})$ defines a strategic-form game where the strategy set of player n is the polytope P_n. Let $C_n^\epsilon = \{p_n \in [0,1]^{L_n} | (\exists \sigma_n \in P_n)(\forall i \in L_n) p_n(i) = \sum_{s\in s_n(i)} \sigma_n(s)\}$. This is the perturbed enabling strategy set used by Govindan and Wilson (2002) to obtain the structure theorem for extensive-form games.

Let $\mathcal{E}^{GW} := \{(G, p) \in \mathcal{G} \times C^\epsilon |\ p$ is an equilibrium of $G\ \}$ be the *Govindan and Wilson equilibrium graph* over payoffs over terminal nodes of the tree (see Govindan and Wilson (2002)). Recall also from Remark 6.9 that the strategic-form payoff function of player n associated to G can be written as $p_n \cdot \nu_n(p_{-n}) + \nu_n(\emptyset)$.

The structure theorem from Govindan and Wilson (2002) allows assignment of degrees (henceforth GW-degrees) to equilibrium components according to the local degrees of $\text{proj}_{\mathcal{G}} : \mathcal{E}^{GW} \to \mathcal{G}$, defined by $\text{proj}_{\mathcal{G}}(G, p) = G$. Therefore, if $D \subset C^\epsilon$ is an equilibrium component of G, we denote the degree of this component by $\deg_G^{GW}(D)$.

Let $r_n : \mathbb{R}^{L_n} \to C_n^\epsilon$ be the nearest-point retraction to C_n^ϵ. Let $r := \times_n r_n$. Consider also the map $\omega_n : C_n^\epsilon \to \mathbb{R}^{L_n}$ defined by $\omega_n(p_n) := p_n + \nu_n(p_{-n})$ with $\omega := \times_n \omega_n$ and let $\Phi_G : \Pi_n C_n^\epsilon \to \Pi_n C_n^\epsilon$, given by $\Phi_G(p) = r \circ \omega$. Lemma 5.1 in Govindan and Wilson (2002) shows that a profile of enabling strategies $(p_n)_{n \in N}$ is an equilibrium of the extensive-form game G with perturbed enabling strategies if and only if it is a fixed point of map Φ_G. The map Φ_G is the analogous version of the GPS-map formulated to the extensive-form game G.

Theorem 6.14. *Let $D \subset C^\epsilon$ be a component of equilibria of the extensive-form game G. Then the fixed point index of D according to Φ_G (denoted $ind(D, \Phi_G))$) equals the degree $deg_G^{GW}(D)$.*

Proof. See Appendix, subsection 7.5. ☐

Proposition 6.15. *Let $D \subset C^\epsilon$ be an equilibrium component of an extensive-form game $G \in \mathcal{G}$. Let V be the strategic-form representation of the extensive-form game G with perturbed enabling strategy sets. Then $deg_V(D) = deg_G^{GW}(D)$.*

Proof. See Appendix, subsection 7.5. ☐

We conclude with a result (Theorem 6.19) establishing the precise relation between the GW-degree theory and normal-form degree theory. Denote by $E(G) \subset \Pi_n C_n^\epsilon$ the set of equilibrium components of extensive-form game G, where the strategy set of each player n is C_n^ϵ. Recall that $E(G)$ is a semi-algebraic set and has, therefore, finitely many connected components. We need a few lemmata to reach the proof of Theorem 6.19. The lemmata demonstrate the usefulness of developing the strategic-form index theory and how it allows to perform the link between GW-degree and normal-form degree.

Recall that we defined the best-reply index of a component of equilibrium D by choosing a suitable neighborhood U of that component that contained no equilibrium in its boundary as well as a continuous function $f : P \to P$ whose graph is "sufficiently close" to the graph of the best-reply correspondence. We can extend the definition of index of a component to the index of a neighborhood – straightforwardly by repeating the reasoning of subsection 3.2.2 – to any open neighborhood in P that does not contain any equilibrium in its boundary. An open neighborhood in P with this property will be called *suitable*. Let U be suitable. We define *the local index of the best-reply at an open neighborhood U* as the fixed point index of $f|_U : U \to P$.

Lemma 6.16. *Let $G := (N, (\Sigma_n)_{n \in N}, (G_n)_{n \in N})$ be a normal-form game and let U be an open neighborhood in Σ that is suitable. Then there exists $\epsilon > 0$ such that for all $n \in N$ and any polytope $P_n \subset int(\Sigma_n)$ with Hausdorff-distance $d(P_n, \Sigma_n) \leq \epsilon$, the open neighborhood $U^\epsilon = U \cap P$ in P has no equilibrium of $G|_P = (N, (P_n)_{n \in N}, G_n|_P)_{n \in N})$ in its boundary in P and the local index of the best-reply of G at U is equal to the local index of the best-reply of $G|_P$ at U^ϵ.*

Proof. See Appendix, subsection 7.5 ☐

Let P_n be the polytope in the definition of C_n^ϵ. Let $p_n : P_n \to C_n^\epsilon$ be defined by $p_n(\sigma_n) = (\sum_{s \in s_n(i)} \sigma_n(s))_{i \in L_n}$. Since p_n is affine and surjective, we can consider its right-inverse $t_n : C_n^\epsilon \to P_n$. Let $p := \times_n p_n$, $t := \times_n t_n$ and $V = (N, (C_n^\epsilon)_{n \in N}, (V_n)_{n \in N})$ be the strategic-form representation of the extensive-form game G, with perturbed enabling strategy set. It then follows immediately that $V_n \circ p = G_n|_P$, so, if C is an equilibrium of $G_n|_P$, then $p(C)$ is an equilibrium of V.

Lemma 6.17. *Let U be a suitable neighborhood in $G|_P$. Then $p(U) = U'$ is suitable for game V and the local index of the best-reply of $G|_P$ at U equals the local index of the best-reply of V at $U^{'}$. In particular, if C is an equilibrium component of game $G|_P$ then $p(C) =: C^{'}$ is an equilibrium component of V and the best-reply indices of C and $C^{'}$ are the same.*

Proof. Apply Theorem 7.13 and Corollary 7.14. □

Corollary 6.18. *Let C be a component of equilibria of the normal-form representation $\tilde{G} = (N, (\Sigma_n)_{n\in N}, (\tilde{G}_n)_{n\in N})$ of an extensive-form game G. Let U be an open neighborhood in Σ of C that contains no other equilibria of $\tilde{G}$ in its boundary. Let $\epsilon > 0$ and U^ϵ be as in Lemma 6.16. Then the best-reply index of C equals the local index of the best-reply of V at $U^{'} = p(U^\epsilon)$, where V is the strategic-form of the extensive-form game G with perturbed strategy set.*

Proof. Notice that $U^{'}$ is an open neighborhood in $C^\epsilon := \times_n C_n^\epsilon$ that has no equilibrium of V in its boundary. The result then follows from immediate application of Lemmata 6.16 and 6.17. □

Theorem 6.19. *Let $G \in \mathcal{G}$ be an extensive-form game and let $\tilde{G} = (N, (\tilde{\Sigma}_n)_{n\in N}, (\tilde{G}_n)_{n\in N})$ be its normal-form representation. If C is an equilibrium component of $\tilde{G}$, let $\epsilon > 0$ and U' be given as in Corollary 6.18. Then*

$$\deg_{\tilde{G}}(C) = \sum_{C' \in E(G)\cap U'} \deg_G^{GW}(C').$$

Proof. See Appendix, subsection 7.5. □

Theorem 6.19 tells us that the normal-form degree of an equilibrium component C can be approximated by certain GW-degrees. The result shows that indeed GW-degrees and normal-form degrees capture essentially the same type of robustness to payoff perturbations. This result is somewhat surprising: although the normal-form perturbations comprise typically a space of much larger dimension than the space of perturbations over terminal payoffs – which is the space with respect to which the GW-degree theory captures perturbations – it turns out that up to a sufficiently close approximation (captured by the sufficiently small ϵ in the statement of the theorem), both theories are essentially equivalent.

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7. Appendix

7.1. **Proofs of Subsection 2.** We first introduce a few necessary concepts in order to prove Theorem 3.1.

Definition 7.1. Let $[\Delta_n]$ be the affine space generated by the unit simplex in $\mathbb{R}^{d_n}$. A polytope $P_n \subset \mathbb{R}^{d_n}$ is called *standard* if:

(1) $P_n \subset [\Delta_n]$,
(2) P_n has dimension $d_n - 1$.

Notice that if $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ is any strategic-form game and P_n has dimension $d_n - 1$, then there exists an affine and bijective mapping $e_n : P_n \to P_n^s \subset \mathbb{R}^{d_n}$ where P_n^s is a standard polytope. Let $e = \times_n e_n$ and $V_n^s = V_n \circ e^{-1}|_{P^s}$, where $P^s = \times_n P_n^s$. Consider the standard strategic-form game $V^s = (N, (P_n^s)_{n\in N}, (V_n^s)_{n\in N})$. The game V^s is the *standard strategic-form game associated to* V.

Because each V_n^s is defined over P^s, each V_n^s is uniquely defined over $\times_n[\Delta_n] \subset \times_n \mathbb{R}^{d_n}$. This implies it has an unique extension to a multilinear functional over $\times_n \mathbb{R}^{d_n}$. Then V_n^s can be represented as vector $(V_n^s(x_1, ..., x_N))_{x_i \in \mathbb{R}^{d_i}, i=1,..,N} \in \mathbb{R}^M$, where $M = d_1 \cdot ... \cdot d_N$ and $x_i \in \mathbb{R}^{d_i}$ denotes a canonical vector of $\mathbb{R}^{d_i}$.

Let $\mathcal{E} = \{(V, \sigma) \in \mathbb{R}^{NM} \times P^s | \sigma \text{ is an equilibrium of } V\}$ be the *graph of equilibria* for the standard strategic-form games where the strategy set of each player is given by standard polytopes P_n^s. Let $\text{proj}_{\mathbb{R}^{NM}} : \mathcal{E} \to \mathbb{R}^{NM}$

be defined by $\text{proj}_{\mathbb{R}^{NM}}(V,p) = V$. We will denote by $V^n(\sigma_{-n})$ the vector $(V_n(x_n,\sigma_{-n}))_{x_n\in\mathbb{R}^{d_n}}$, where

$$V_n(x_n,\sigma_{-n}) := \sum_{x_j\in\mathbb{R}^{d_j}:j\neq n} V_n(x_n,x_{-n})\Pi_{j\neq n}\sigma_{jx_j},$$

where $\sigma_j = (\sigma_{jx_j})_{x_j\in\mathbb{R}^{d_j}} \in \mathbb{R}^{d_j}$.

Remark 7.2. Given a set X, we denote by $id_X : X \to X$ the identity function on X. Whenever the space X is implicitly defined, we use id. If $x \in \mathbb{R}^n$, then $||x||_\infty := \sup\{|x_j| : x = (x_1,...,x_n)\}$. The symbol $\mathbb{S}^{NM}$ denotes the topological space $\mathbb{R}^{NM} \cup \{\infty\}$ with the one-point compactification topology.

We first prove the Structure Theorem for standard strategic-form games.

Theorem 7.3. *There exists a homeomorphism $\theta : \mathcal{E} \to \mathbb{R}^{NM}$ such that there is a linear homotopy between $\text{proj}_{\mathbb{R}^{NM}} \circ \theta^{-1}$ and the identity function on $\mathbb{R}^{NM}$ and this homotopy extends to a homotopy on the one-point compactification of $\mathbb{R}^{NM}$.*

Proof. Let σ^*_n be a uniform distribution over the vertices of the polytope P^s_n and $V \in \mathbb{R}^{NM}$. Defining $g_{nx_n} = V_n(x_n,\sigma^*_{-n})$, we have that $V_n(x_1,...,x_N) = \bar{V}_n(x_1,..,x_N) + g_{nx_n}$, where $\bar{V}_n(x_1,...,x_N) := V_n(x_1,...,x_N) - g_{nx_n}$. The decomposition of V in $\bar{V}$ plus g is unique and we denote it by $V = \bar{V} \bigoplus g$.

Let $\sigma \in \times_n P^s_n$ be an equilibrium of the strategic-form game V and $r_n : \mathbb{R}^{d_n} \to P^s_n$ the nearest-point retraction. Define $z_{nx_n} := \bar{V}_n(x_n,\sigma_{-n}) + g_{nx_n} + \sigma_{nx_n}$. Then $r_{nx_n}(z_n) = \sigma_{nx_n}$. Indeed, the variational inequality below characterizes a unique $r_n(z_n)$:

$$\langle \tau_n - r_n(z_n), z_n - r_n(z_n)\rangle \le 0, \forall \tau_n \in P^s_n. \tag{6}$$

Then, if σ is an equilibrium, by definition it implies $\langle \tau_n - \sigma_n, V^n(\sigma_{-n})\rangle \le 0$, for all $\tau_n \in P^s_n$. Rewriting this inequality $\langle \tau_n - \sigma_n, \bar{V}^n(\sigma_{-n}) + g_n + \sigma_n - \sigma_n\rangle \le 0$ shows $r_n(z_n) = \sigma_n$.

Using the decomposition of $V = \bar{V} \bigoplus g$, we define a mapping θ from the equilibrium graph $\mathcal{E} = \{(V,\sigma) \in \mathbb{R}^{NM} \times P^s | \sigma \text{ is an equilibrium of } V\ \}$ to $\mathbb{R}^{NM}$ by $\theta(\bar{V},g,\sigma) = (\bar{V},z)$. We show that θ is a homeomorphism. First, θ is clearly continuous. Also, the inverse homeomorphism can be defined explicitly: $h : \mathbb{R}^{NM} \to \mathcal{E}$ with $h(\bar{V},z) = (\bar{V},g,r(z))$, where $g := (g_1,..,g_N)$, $g_j := z_j - \sigma_j - \bar{V}^j(\sigma_{-j})$ and $\sigma_j := r_j(z_j), \forall j \in N$. It follows that $h \circ \theta = id_{\mathcal{E}}$ and $\theta \circ h = id_{\mathbb{R}^{NM}}$.

Let $\mathbb{S}^{NM}$ denote the NM-dimensional sphere and recall that $\mathbb{S}^{NM}$ is homeomorphic to $\mathbb{R}^{NM} \cup \{\infty\}$ with the one-point compactification topology. Let $\overline{\mathcal{E}}$ denote the one-point compactification of $\mathcal{E}$.

We define a homotopy $H : [0,1] \times \mathbb{S}^{NM} \to \mathbb{S}^{NM}$ by $H(t,V) = H(t,\bar{V},z) = (\bar{V}, tz + (1-t)g)$, if $V \in \mathbb{R}^{NM}$, and $H(t,\infty) = \infty$. Since $\text{proj}_{\mathbb{R}^{NM}}$ and h are both continuous and proper mappings, they have continuous extensions $\overline{\text{proj}}_{\mathbb{R}^{NM}} : \overline{\mathcal{E}} \to \mathbb{S}^{NM}$ and $\overline{h} : \mathbb{S}^{NM} \to \overline{\mathcal{E}}$ to the one-point compactifications, both taking ∞ to ∞. Notice now that $H(0,\cdot) = \overline{\text{proj}}_{\mathbb{R}^{NM}} \circ \overline{h} : \mathbb{S}^{NM} \to \mathbb{S}^{NM}$ and $H(1,\cdot) = id_{\mathbb{S}^{NM}}$. We now show that H is continuous, which shows H is indeed a homotopy. Continuity at points (t,V) where $V \neq \infty$ is immediate from the definition, since the homotopy is linear. It remains to show the continuity of H at all points $[0,1] \times \{\infty\}$ or equivalently $\forall R > 0$, $\exists S > 0$ such that if $||(\bar{V},z)|| \ge S$, it implies $\forall t$, $||H(t,\bar{V},z)|| \ge R$.

Note that the definition of g implies that $|z_{nx_n} - g_{nx_n}| \le |\sigma_{nx_n}| + |\bar{V}_n(x_n,\sigma_{-n})|$, where for all n $\sigma_{nx_n} := r_{nx_n}(z_n)$. Because $\sigma_{nx_n} = \text{sign}(\sigma_{nx_n})|\sigma_{nx_n}|$, it implies

$$\bar{V}_n(x_n,\sigma_{-n}) = \sum_{x_j:j\neq n} \bar{V}_n(x_n,x_{-n})\Pi_{j\neq n}\sigma_{jx_j} = \sum_{x_j:j\neq n} \bar{V}_n(x_n,x_{-n})h_n(x_{-n})\Pi_{j\neq n}|\sigma_{jx_j}|$$

with $h_n(x_{-n}) = \Pi_{j\neq n}\text{sign}(\sigma_{jx_j})$, which implies that $|h_n(x_{-n})| = 1$. Let $||\Pi_{j\neq n}|\sigma_j|||_\infty := \sup\{\Pi_{j\neq n}|\sigma_{jx_j}| : j \in N, x_j \in \mathbb{R}^{d_j}\}$. Because P^s_j is a polytope, there exists $\alpha > 0$ such that for all $\sigma_j \in P^s_j$, we have $||\Pi_{j\neq n}|\sigma_j|||_\infty \leq \alpha$, for all n. We can assume without loss of generality $\alpha > 1$. Also, there exists $B > 1$ such that for all $j \in N$ and $\sigma_j \in P^s_j$ it implies $||\sigma_j||_\infty \leq B$. Therefore, we have that for some $C > 1$:

$$||z_n - g_n||_\infty \leq ||\bar{V}||_\infty C\alpha + B. \tag{7}$$

Let $R > 0$. Set $S = 2RC\alpha + B$. If $||(\bar{V}, z)||_\infty \geq S$, then either $||\bar{V}||_\infty \geq R$, in which case $||H(t, \bar{V}, z)||_\infty \geq R$, or $||\bar{V}||_\infty < R$ and $||z||_\infty \geq 2RC\alpha + B$. Using 7 we have $||tz + (1-t)g||_\infty \geq R$. □

Let P_n be an arbitrary polytope and P^s_n a standard polytope such that there exists an affine bijection $e_n : P_n \to P^s_n$ with $P := \Pi_n P_n$. Let $e := \times_n e_n$ and $\bar{e} : A(P) \to \mathbb{R}^M$ defined as follows: for $V_n \in A(P)$, define $V^s_n = (V_n \circ e^{-1}) : P^s \to \mathbb{R}$. By construction, V^s_n can be uniquely extended to a unique multilinear functional $V^s_n : \times_n \mathbb{R}^{d_n} \to \mathbb{R}$. We define $\bar{e}(V_n) = V^s_n$. It is easy to check that $\bar{e}$ is a linear isomorphism. Therefore, for $V \in \mathbb{R}^{NM}$ the mapping $T := \times^N_{n=1}\bar{e} : \Pi_n A(P) \to \mathbb{R}^{NM}$ is also a linear isomorphism. Hence the extension of T to the one point-compactifications of $\Pi_n A(P)$ and $\mathbb{R}^{NM}$ is a homeomorphism and provides an orientation to the one-point compactification $\overline{\Pi_n A(P)}$.

Proof of Theorem 3.1. Define the following mapping $e^* : \mathcal{E}^* \to \mathcal{E}$ by $e^*(V, p) = (T(V), e(p))$. The map e^* is a homeomorphism. This implies that there exists a homeomorphism between $\mathcal{E}^*$ and $\Pi_n A(P)$ given by $\theta^* = T^{-1} \circ \theta \circ e^* : \mathcal{E}^* \to \Pi_n A(P)$. Let now $q : [0,1] \times \Pi_n A(P) \to \Pi_n A(P)$ be defined by $q(t, V) = T^{-1} \circ H(t, T(V))$, where H is the homotopy from Theorem 7.3. Since T is a proper mapping, it follows that the homotopy q also extends to a homotopy in the one-point compactification of $\Pi_n A(P)$ continuously. Notice now that $\text{proj}^* \circ (e^*)^{-1} = T^{-1} \circ \text{proj}_{\mathbb{R}^{NM}}$ and from the definition of θ^* we have that $(\theta^*)^{-1} = (e^*)^{-1} \circ \theta^{-1} \circ T$. Combining these two facts, it implies $\text{proj}^* \circ (\theta^*)^{-1} = \text{proj}^* \circ ((e^*)^{-1} \circ \theta^{-1} \circ T) = T^{-1} \circ \text{proj}_{\mathbb{R}^{NM}} \circ \theta^{-1} \circ T$. This in turn implies that $q(0, \cdot) = \text{proj}^* \circ (\theta^*)^{-1}$ and $q(1, \cdot) = id_{\Pi_n A(P)}$. □

The mapping $(\theta^*)^{-1}$ provides $\mathcal{E}^*$ with an orientation and proj^* has degree $+1$ according to this orientation. This allows us to define degrees of equilibrium components for arbitrary strategic-form games. Corollary 7.4 shows that the degree is invariant under any embedding e and associated map T. This Corollary will be used in other results further ahead.

Corollary 7.4. *Let C be a component of equilibrium of game $V \in \Pi_n A(P)$. Then the $\deg_V(C) = \deg_{T(V)}(e(C))$.*

Proof. Let C be a component of equilibrium of game $V \in \Pi_n A(P)$. Then $e(C) \subset P^s$ is an equilibrium component of $T(V)$. Let W be an open neighborhood in the graph $\mathcal{E}^*$ of $\{V\} \times C =: F$ such that the closure $\text{cl}(W)$ contains no other pair $(V, p) \in \mathcal{E}^*$. Then it follows by definition that the degree of C of game V is the local degree of the mapping $\text{proj}^*|_W : W \to \mathbb{S}^{NM}$ over V. Let $U := e^*(W)$, which is open because e^* is a homeomorphism. Notice that $\{T(V)\} \times e(C) =: K$ is a compact subset of U such that $\text{cl}(U)$ contains no pair $(T(V), p) \in \mathcal{E}$ in its boundary. Therefore, the degree of $e(C)$ of game $T(V)$ is the local degree of $\text{proj}|_U : U \to \mathbb{S}^{NM}$ over $T(V)$. Notice now that $T \circ \text{proj}^*|_W = \text{proj}|_U \circ e^*|_W$. Notice that map T has degree $+1$ by definition. The map e^* also has degree $+1$: indeed, the first diagram below commutes since $e^* \circ \theta^{-1} = (\theta^*)^{-1} \circ T^{-1}$ - which implies the first square commutes by functoriality - and the naturality of the long exact sequence in homology - which implies the second square of the diagram commutes:

$$\begin{array}{ccccc} H_{NM}(\mathbb{S}^{NM}) & \xrightarrow{(\theta)_*^{-1}} & H_{NM}(\bar{\mathcal{E}}) & \xrightarrow{j_*} & H_{NM}(\bar{\mathcal{E}}, \bar{\mathcal{E}} - K) \\ \downarrow T_*^{-1} & & \downarrow (e^*)_*^{-1} & & \downarrow (e^*)_*^{-1} \\ H_{NM}(\overline{\Pi_n A(P)}) & \xrightarrow{(\theta^*)_*^{-1}} & H_{NM}(\bar{\mathcal{E}}^*) & \xrightarrow{j_*} & H_{NM}(\bar{\mathcal{E}}^*, \bar{\mathcal{E}}^* - F) \end{array}$$

The map j denotes an inclusion. This in particular shows that $(e^*)^{-1} : (\bar{\mathcal{E}}, \bar{\mathcal{E}} - K) \to (\bar{\mathcal{E}}^*, \bar{\mathcal{E}}^* - F)$ has degree +1. It follows that $(e^*)^{-1}|_{(U,U-K)} : (U, U - K) \to (W, W - F)$ also has degree +1. Consider finally the commutative diagram - which commutes by functoriality of homology:

$$\begin{array}{ccc} H_{NM}(U, U - K) & \xrightarrow{(\mathrm{proj}_{\mathbb{R}^{NM}})_*} & H_{NM}(\mathbb{S}^{NM}, \mathbb{S}^{NM} - \{T(V)\}) \\ \downarrow ((e^*)^{-1}|_{U,U-K})_* & & \downarrow (T)_*^{-1} \\ H_{NM}(W, W - F) & \xrightarrow{(proj^*)_*} & H_{NM}(\overline{\Pi_n A(P)}, \overline{\Pi_n A(P)} - \{V\}) \end{array}$$

Because degree of $(e^*)^{-1}|_{(U,U-K)}$ as well as degree of $(T)^{-1}|_{(\mathbb{S}^{NM}, \mathbb{S}^{NM} - \{T(V)\})}$ are +1, commutation of the diagram yields that $\deg_V(C) = \deg_{T(V)}(e(C))$. □

Proof of Proposition 3.4. It is clear that $E(C)$ is an equilibrium of V', by definition of payoffs in V'. We prove the equality between degrees. Let $e_n : P_n \to P_n^s$ be an affine bijection between the arbitrary polytope P_n and a standard polytope P_n^s and $e := \times_n e_n$. Then $e_n' = e_n \circ E_n^{-1} : P_n' \to P_n^s$ is also an affine bijection. Let $e' := \times_n e_n'$. By Corollary 7.4, it follows that $\deg_V(C) = \deg_{T(V)}(e(C))$, where $T_n(V_n) := V_n \circ e^{-1}$ and $T := \times_n T_n$. Now define $T_n' : A(P') \to A(P^s)$ by $T_n'(\tilde{V}_n) = \tilde{V}_n \circ (e')^{-1}$ and let $T' := \times_n T_n'$. Notice that $T(V) = T'(V')$ and $e'(E(C)) = e(C)$. This implies therefore, by Corollary 7.4, that $\deg_{T(V)}(e(C)) = \deg_{V'}(E(C))$, which concludes the proof. □

Proof of Proposition 3.5. The map $\mathrm{proj}^*|_U$ is a proper map. Now apply Proposition 5.12 in Chapter IV of Dold (2012) to obtain that there exists a sufficiently small open neighborhood W of V such that for each $V' \in W$ the local degree $\mathrm{proj}^*|_U$ over V' is well defined and is equal to $\deg_V(C)$. Therefore the local degree $\mathrm{proj}^*|_U$ over $V' \in W$ is nonzero. This implies, by definition of the degree, that for any $V' \in W$, there exists an equilibrium p' of V' with $(V', p') \in U$. □

7.2. **Proofs of Subsection 3.2.** The proof of Theorem 3.11 is performed in steps. First, Lemma 7.6 establishes the result for standard strategic-form games. Then the proof of Theorem 3.11 is presented using this lemma.

Let $\mathcal{E} = \{(V, \sigma) \in \mathbb{R}^{NM} \times P^s | \sigma \text{ is an equilibrium of } V\}$ be the graph of equilibria of **standard strategic-form games** – see Definition 7.1 – with standard strategy polytope P_n^s for each player $n \in N$. Recall that $\mathcal{E}(V)$ denotes the set of equilibria of V.

Remark 7.5. Given a Nash-map $f : \mathbb{R}^{NM} \times P^s \to P^s$, we abuse notation slightly and use f to denote the extension $f \circ (id_{\mathbb{R}^{NM}} \times r)$, where $r := \times_n r_n$ and $r_n : \mathbb{R}^{d_n} \to P_n^s$ is the nearest-point retraction.

Let $P_n^\epsilon \subset [\Delta_n]$ be a polytope containing P_n^s in its relative interior. Let $\Delta = \{p \in F | (\forall n \in N), p_n \in P_n^\epsilon\}$. Denote by $\partial\Delta$ the topological boundary of Δ in F. We view the graph $\mathcal{E}$ as a subset of $\mathbb{R}^{NM} \times \Delta$. The proof of Lemma 7.6 is an adaptation of an unpublished proof of Govindan and Wilson. The result for normal form games is proved in Demichelis and Germano (2002).

Lemma 7.6. *Let f^1 and f^2 be two Nash-maps. Then, the two displacement maps d^1 and d^2 are homotopic as maps between the triads $(\mathbb{R}^{NM} \times \Delta, \mathcal{E}, (\mathbb{R}^{NM} \times \Delta) - \mathcal{E})$ and $(F_0, 0, F_0 - \{0\})$. Consequently, for every standard strategic-form game V and every component C, $ind(C, f^1) = ind(C, f^2)$.*

Proof. We start by proving a claim:

Claim 7.7. $d^1 : \mathbb{R}^{NM} \times \partial\Delta \to F_0 - \{0\}$ *is homotopic to* $d^2 : \mathbb{R}^{NM} \times \partial\Delta \to F_0 - \{0\}$.

Proof. Since f^1 and f^2 map $\mathbb{R}^{NM} \times \partial\Delta$ into P^s, d^1 and d^2 are homotopic via the linear homotopy. □

Claim 7.8. $\mathbb{R}^{NM} \times \partial\Delta$ *is a deformation retract of* $(\mathbb{R}^{NM} \times \Delta) - \mathcal{E}$

Proof. As in the Structure Theorem proved in Theorem 7.3, we reparametrize the space of games as $V = (\bar{V}, g)$. Let Γ_0 be the linear subspace of payoffs vectors containing $\bar{V}$. Then $\mathbb{R}^{NM} = \Gamma_0 \times \mathbb{R}^m$, where $m = d_1 + ... + d_n$. Define the function h from $\mathbb{R}^{NM} \times \Delta$ to itself by $h(\bar{V}, g, p) = (\bar{V}, z, p)$, where

$$z_{nx_n} = p_{nx_n} + V_n(x_n, p_{-n}),$$

where we maintain the notation of subsection 2. It easily follows that h is a homeomorphism that maps $\mathbb{R}^{NM} \times \partial\Delta$ onto itself. Let $q : \mathbb{R}^m \to P^s$ be the nearest-point retraction. Denoting Q the graph of q, we have that $h(\mathcal{E}) = \Gamma_0 \times Q$. It is enough to prove therefore that $\mathbb{R}^{NM} \times \partial\Delta$ is a deformation retract of $(\mathbb{R}^{NM} \times \Delta) - (\Gamma_0 \times Q)$. We can construct a retraction ψ of $(\mathbb{R}^{NM} \times \Delta) - (\Gamma_0 \times Q)$ onto $\mathbb{R}^{NM} \times \partial\Delta$ as follows. First, given a pair $(z, p) \in (\mathbb{R}^m \times \Delta) - Q$, let $\eta(z, p)$ be the unique point in $\partial\Delta$ that lies on the ray emanating from $q(z)$ and passing through p. Then define $\psi(\bar{V}, z, p) = (\bar{V}, z, \eta(z, p))$. The map ψ is easily seen to be a retraction. Let $i_Q : \mathbb{R}^{NM} \times \partial\Delta \to (\mathbb{R}^{NM} \times \Delta) - (\Gamma_0 \times Q)$ be the inclusion map. Then $i_Q \circ \psi$ is homotopic to the identity map using the linear homotopy. Therefore, $\mathbb{R}^{NM} \times \partial\Delta$ is a deformation retract of $(\mathbb{R}^{NM} \times \Delta) - (\Gamma_0 \times Q)$. □

Claim 7.9. $d^1 : (\mathbb{R}^{NM} \times \Delta) - \mathcal{E} \to F_0 - \{0\}$ *is homotopic to* $d^2 : (\mathbb{R}^{NM} \times \Delta) - \mathcal{E} \to F_0 - \{0\}$.

Proof. Let id be the identity map on $(\mathbb{R}^{NM} \times \Delta) - \mathcal{E}$ and let $j_{\mathcal{E}} : \mathbb{R}^{NM} \times \partial\Delta \subset (\mathbb{R}^{NM} \times \Delta) - \mathcal{E}$. By Claim 7.8, there exists a retraction ϕ from $(\mathbb{R}^{NM} \times \Delta) - \mathcal{E}$ to $\mathbb{R}^{NM} \times \partial\Delta$ such that id is homotopic to $j_{\mathcal{E}} \circ \phi$. Therefore, for $i = 1, 2$, $(d^i \circ id)$ is homotopic to $(d^i \circ j_{\mathcal{E}} \circ \phi) : (\mathbb{R}^{NM} \times \Delta) - \mathcal{E} \to F_0 - \{0\}$. By Claim 7.7, the restrictions of d^1 and d^2 to $\mathbb{R}^{NM} \times \partial\Delta$ are homotopic. Therefore, d^1 is homotopic to $d^2 : (\mathbb{R}^{NM} \times \Delta) - \mathcal{E} \to F_0 - \{0\}$. □

We now construct the homotopy of Lemma 7.6. Let Φ be a homotopy between the restrictions of d^1 and d^2 to $(\mathbb{R}^{NM} \times \Delta) - \mathcal{E}$. It is readily checked from the constructions above that $\Phi([0, 1] \times (\mathbb{R}^{NM} \times \Delta) - \mathcal{E})$ is a bounded subset of F_0. By Urysohn's Lemma, there exists a continuous function $\alpha : \mathbb{R}^{NM} \times \Delta \to [0, 1]$ such that $\alpha^{-1}(0) = \mathcal{E}$. Define then $D : (\mathbb{R}^{NM} \times \Delta) - \mathcal{E} \times [0, 1] \to F_0 - \{0\}$ by $D(x, t) = \alpha(x)\Phi(x, t)$. The image of Φ being bounded, D has a continuous extension to a map from $\mathbb{R}^{NM} \times \Delta$ to F_0 that maps $\mathcal{E}$ to 0. The result then follows from the observation that for $i = 1, 2$, d^i is homotopic to $D(\cdot, i - 1)$. □

Proof of Theorem 3.11. Let P_n^s be a standard polytope such that there exists an affine bijection $e_n : P_n \to P_n^s$ and let $e := \times_n e_n$. Let $T_n : A(P) \to \mathbb{R}^M$ be defined as in subsection 7.1 and $T := \times_n T_n$. Let $f_i' := e \circ f_i \circ (T^{-1} \times e^{-1}) : \mathbb{R}^{NM} \times P^s \to P^s, i = 1, 2$. Then f_i' is a Nash-map and Lemma 7.6 shows that the indices of equilibrium component $e(C)$ according to f'_1 and f'_2 are the same. Let V^s be a standard strategic-form game such that $T^{-1}(V^s) = V$. Considering the restriction $(f'_i)_{V^s} : P^s \to P^s$ we have that $(f'_i)_{V^s} = e \circ (f_i)_V \circ e^{-1}$, by definition. Now define $h_i := (f_i)_V \circ e^{-1}$. The commutativity property of the index (see Dold (2012)) shows that the fixed-point set of $h_i \circ e$ and $e \circ h_i$ are homeomorphic and the index of each fixed-point component is the same under these two maps. Since $e \circ h_i = (f'_i)_{V^s}$ and $h_i \circ e = (f_i)_V$, this shows that the index of C under $(f_i)_V$ is the same as the index of $e(C)$ under $(f'_i)_{V^s}$. Since i is arbitrary, it follows therefore that $\mathrm{ind}(C, f_1) = \mathrm{ind}(C, f_2)$. □

Proof of Proposition 3.12. Let $T : \Pi_n A(P) \to \Pi_n A(P')$ be defined by $T_n(V_n) = V_n \circ E^{-1}$ and $T := \times_n T_n$. Then $f = E^{-1} \circ f_2 \circ (T \times E)$ is a Nash-map. By Theorem 3.11, $\mathrm{ind}(C, f) = \mathrm{ind}(C, f_1)$. Now, by the commutativity property of the index (see Dold (2012)), it follows that $\mathrm{ind}(C, f) = \mathrm{ind}(E(C), f_2)$. □

7.3. **Proofs of Section 4.** We start by proving Theorem 4.1. The proof requires a few lemmata and definitions.

Let $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ be a standard strategic-form game. Let $\mathrm{w}^n_V : P \to \mathbb{R}^{d_n}$ be given by $\mathrm{w}^n_V(\sigma) = p_n + V^n(\sigma_{-n}) \in \mathbb{R}^{d_n}$. Given the variational inequality 6 above, we have that defining $r := \times_n r_n$, $\mathrm{w}_V := \times_n \mathrm{w}^n_V$ and $\Phi_V := r \circ \mathrm{w}_V : P \to \Pi_j \mathbb{R}^{m_j} \to P$, σ is an equilibrium of V if and only if it is a fixed point of Φ_V.

The idea of the GPS-map is to use the variational inequality 6 that characterizes the nearest-point retraction: if $p \in P$ is a fixed-point of Φ_V, then p satisfies $\forall n \in N$, $\langle p'_n - p_n, p_n + V^n(p_{-n}) - p_n\rangle \leq 0, \forall p'_n \in P_n$ which implies $\langle p'_n - p_n, V^n(p_{-n})\rangle \leq 0$. Now, when the game is standard, $p_n \cdot V^n(p_{-n})$ is precisely the payoff to player n, and $\langle p'_n - p_n, V^n(p_{-n})\rangle \leq 0$ shows that p_n is indeed a best-reply and that p is an equilibrium.

The commutativity property of the index (see Dold (2012), Chapter VII, Theorem 5.14) gives us that the sets of fixed points of Φ_V and of the permuted map $F = \mathrm{w}_V \circ r$ are homeomorphic and their indices agree. In particular, by definition, the index is the local degree (over 0) of the displacement map $f := id - F$. We denote the *GPS-index of an equilibrium component C of game V* as $\mathrm{ind}_{GPS-V}(C)$.

Lemma 7.10. *Let $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ be a standard strategic-form game and let C be a component of equilibrium of V. Then $\deg_V(C) = \mathrm{Ind}_{GPS-V}(C)$.*

Proof. Let $g \in \Pi_n \mathbb{R}^{d_n}$. Consider a strategic-form game $V \bigoplus g$ whose payoff function is defined by $(V \bigoplus g)_n(\sigma) = \sigma_n \cdot V^n(\sigma_{-n}) + \sigma_n \cdot g_n$ and player n strategy set is P_n. Let $\mathcal{E}_V = \{(g, \sigma) \in \Pi_i \mathbb{R}^{d_i} \times P | \sigma$ is an equilibrium of $V \bigoplus g\}$ be the graph of equilibria over the restricted class of perturbations g. Define $\Theta : \mathcal{E}_V \to \Pi_n \mathbb{R}^{d_n}$ by its coordinate functions $\Theta_n(g, \sigma) = \sigma_n + V^n(\sigma_{-n}) + g_n \in \mathbb{R}^{d_n}$ and $\mathrm{proj}_g : \mathcal{E}_V \to \Pi_i \mathbb{R}^{d_i}$ be the projection over the first coordinate. Then Θ is a homeomorphism and $\Theta^{-1}(z) = (f(z), r(z))$. Therefore $\mathrm{proj}_g \circ \Theta^{-1} = f$. Similarly as in the previous subsection, Θ allows us to provide an orientation to the one point-compactification $\overline{\mathcal{E}_V}$ according to which the degree of $\mathrm{proj}_g : \mathcal{E}_V \to \mathbb{R}^{NM}$ is +1. Similarly to the definition of the degree of an equilibrium component we presented in section 3, we can assign degrees to equilibrium components in the following way: let C be an equilibrium component of V and consider U an open neighborhood in $\mathcal{E}_V$ containing $\{0\} \times C$ and no other pair $(0, \sigma) \in \mathrm{cl}(U)$. Then the local degree of $\mathrm{proj}_g|_U$ over $\{0\}$ is well defined. If we consider now $\Theta(U) = W$, since $(\mathrm{proj}_g|_U \circ \Theta^{-1})|_W = f|_W$, the local degree of $(\mathrm{proj}_g|_U \circ \Theta^{-1})|_W$ over $\{0\}$ equals the $\mathrm{Ind}_{GPS-V}(C)$.

Now we show that the degree of an equilibrium component of V calculated with respect to proj_g equals the degree of the same component but calculated with respect to $\text{proj}_{\mathbb{R}^{NM}}$ – that is, the degree of the component calculated with respect to proj_g agrees with our original definition of degree.

As it happens in the proof of Theorem 7.3, we can decompose V and write $V = (\tilde{V}, g)$. Let $\mathcal{E}_{\tilde{V}} = \{(g', \sigma) | ((\tilde{V}, g'), \sigma) \in \mathcal{E}\}$. Notice that the homeomorphism θ in Theorem 7.3 is the identity in the first coordinate. This allows us to define another homeomorphism $\tilde{\Theta} : \mathcal{E}_{\tilde{V}} \to \Pi_i \mathbb{R}^{d_i}$ by $\tilde{\Theta}(g', \sigma) = z$, where z satisfies $\theta(\tilde{V}, g', \sigma) = (\tilde{V}, z)$. As we did above, we can also assign a degree to the component C as the local degree over g of the projection map $\text{proj}_{g'}$ from an open neighborhood O of $\{g\} \times C$ in $\mathcal{E}_{\tilde{V}}$ whose closure does not contain any other point (g, p).

First, let U be an open neighborhood in $\mathcal{E}$ of $(\tilde{V}, g, C) \in \mathcal{E}$ such that $\text{cl}(U)$ has no other point $(\tilde{V}, g, p)$. Consider $\theta(U)$, where θ is the homeomorphism defined in Theorem 7.3. There exist U_1 an open neighborhood of $\tilde{V}$ and U_2 an open neighborhood $\tilde{\Theta}(g, C)$ such that $U_1 \times U_2 \subset \theta(U)$. The local degree of the equilibrium component C of game $(\tilde{V}, g)$ can therefore be calculated with respect to the map $\text{proj}_{\mathbb{R}^{NM}} \circ \theta^{-1}|_{U_1 \times U_2} : U_1 \times U_2 \to \mathbb{S}^{NM}$ - according to Proposition 5.5, Chapter IV in Dold (2012). Now consider the map $(id \times \text{proj}_{g'} \circ (\tilde{\Theta})^{-1})|_{U_1 \times U_2} : U_1 \times U_2 \to \mathbb{S}^{NM}$ defined by $(id \times \text{proj}_{g'} \circ (\tilde{\Theta})^{-1})(\tilde{V}, z) = (\tilde{V}, \text{proj}_{g'} \circ (\tilde{\Theta})^{-1}(z))$.

Recall now that the homeomorphism θ in Theorem 7.3 is the identity in the first coordinate: $\theta(\tilde{V}, g, \sigma) = (\tilde{V}, \theta_2(\tilde{V}, g, \sigma))$. We have therefore that $\text{proj}_{\mathbb{R}^{NM}} \circ \theta^{-1}(\tilde{V}', z') = (\tilde{V}', \theta_2^{-1}(\tilde{V}', z'))$, where $\theta^{-1}(\tilde{V}', z') = (\tilde{V}', \theta_2^{-1}(\tilde{V}', z'), r(z'))$ and similarly $(id \times \text{proj}_{g'} \circ (\tilde{\Theta})^{-1}))(\tilde{V}', z') = (\tilde{V}', \theta_2^{-1}(\tilde{V}, z'))$. Notice that in the above expression of $id \times \text{proj}_{g'} \circ (\tilde{\Theta})^{-1}$ we have that the second coordinate function θ_2^{-1} fixes the argument $\tilde{V}$.

Let $H : [0, 1] \times U_1 \times U_2 \to \mathbb{R}^{NM}$ be defined by $H(t, \tilde{V}', z') := (\tilde{V}', t\theta_2^{-1}(\tilde{V}', z') + (1-t)\theta_2^{-1}(\tilde{V}, z'))$. By the homotopy property of the degree (see Brown (2014), Theorem 9.5), it follows that the local degree over V of $id \times \text{proj}_{g'} \circ (\tilde{\Theta})^{-1}|_{U_1 \times U_2}$ and $\text{proj}_{\mathbb{R}^{NM}} \circ \theta^{-1}|_{U_1 \times U_2}$ is the same. Finally, Theorem 9.7 in Brown (2014) implies that degree of C with respect to the map $id \times \text{proj}_{g'} \circ (\tilde{\Theta})^{-1}|_{U_1 \times U_2}$ equals the degree of C with respect to the map $\text{proj}_{g'} \circ (\tilde{\Theta})^{-1}|_{U_2}$. Hence $\deg_V(C)$ is equal to the local degree of $\text{proj}_{g'} \circ (\tilde{\Theta})^{-1}|_{U_2}$.

We now finish the proof by showing that the degree of C according to $\text{proj}_{g'} \circ \tilde{\Theta}^{-1}$ agrees with the definition according to $\text{proj}_g \circ \Theta^{-1}$.

Fix now W an open neighborhood of $\tilde{\Theta}(\{g\} \times C)$ such that $W \subset U_2$. Let $f' := \text{proj}_{g'} \circ (\tilde{\Theta})^{-1}|_W$. We have that $\tilde{\Theta}(\{g\} \times C) = \Theta(\{0\} \times C)$ and $f = f' - g$. Let $g : \mathbb{S}^n \to \mathbb{S}^n$ defined by $g(y) = y - g$, if $y \in \mathbb{R}^n$ and $g(\infty) = \infty$. Then $g(f'(x)) = f'(x) - g = f(x)$. Since the degree of $g_* : H_{NM}(\mathbb{R}^{NM}, \mathbb{R}^{NM} - \{g\}) \to H_{NM}(\mathbb{R}^{NM}, \mathbb{R}^{NM} - \{0\})$ is $+1$, it follows that the local degree of $f|_W$ over $\{0\}$ equals the local degree of $f'|_W$ over $\{g\}$. Finally, by construction, the local degree under the map $f|_W$ over 0 is the index of the component $w_V(C)$ of the fixed-point set of F, which is the same as the index of C under the GPS-map Φ, by the commutativity property. Thus $\deg_V(C) = \text{Ind}_{GPS-V}(C)$. □

Proposition 7.11. *Let C be a component of equilibria of a standard strategic-form game $V = (N, (P_n)_{n \in N}, (V_n)_{n \in N})$. Then $\text{Ind}_{BR^V}(C) = \text{Ind}_{GPS-V}(C)$, where BR^V is the best-reply of V.*

Proof. Let $V(P_n)$ be the set of vertices of P_n and let $N_{P_n}(v_n)$ be the normal cone to P_n at $v_n \in V_n(P_n)$, defined by $N_{P_n}(v_n) := \{d \in \mathbb{R}^{d_n} | d \cdot (v_j - v_n) \leq 0, \forall v_j \in V(P_n)\}$. The union of the normal cones over the vertices generates $\mathbb{R}^{d_n}$ and induces a polyhedral subdivision of $\mathbb{R}^{d_n}$ called the normal fan of polytope P_n. The cells of this subdivision are the normal cones (see Ziegler (2012), p.206).

Claim 7.12. *Fix $p \in P$ and $n \in N$. There exists $\lambda_0 > 0$ such that $\forall \lambda > \lambda_0$ the nearest-point retraction $r_n(p_n + \lambda V^n(p_{-n})) \in BR_n^V(p_{-n})$.*

Notice first that $V^n(p_{-n})$ is the gradient of the affine function $f_n : \mathbb{R}^{d_n} \to \mathbb{R}$ defined by $f_n(p_n) = p_n \cdot V^n(p_{-n})$. If $v_n \in V_n(P_n)$ is a maximum for $\max_{p_n \in P_n} f_n(p_n)$, v_n can be characterized by the following "first-order" condition:

$$v_n \in \operatorname{argmax}_{p_n \in P_n} f_n(p_n) = BR_n^V(p_{-n}) \iff V^n(p_{-n}) \in N_{P_n}(v_n) \tag{8}$$

Because the union of the normal cones at the vertices of P_n is $\mathbb{R}^{d_n}$, there exists $\tilde{v} \in V(P_n)$ such that $V^n(p_{-n}) \in N_{P_n}(\tilde{v})$. Assume first that $V^n(p_{-n}) \in \operatorname{int}(N_{P_n}(\tilde{v}))$. Then for $\lambda > 0$ sufficiently large $\frac{p_n - \tilde{v}}{\lambda} + V^n(p_{-n}) \in N_{P_n}(\tilde{v})$. This implies that $\lambda(\frac{p_n - \tilde{v}}{\lambda} + V^n(p_{-n})) = p_n - \tilde{v} + \lambda V^n(p_{-n}) \in N_{P_n}(\tilde{v})$, which implies by definition of the normal cone $\langle p_n + \lambda V^n(p_{-n}) - \tilde{v}, p' - \tilde{v}\rangle \leq 0, \forall p' \in P_n$, which is the inequality that characterizes the nearest-point retraction. Therefore, $r_n(p_n + \lambda V^n(p_{-n})) = \tilde{v} \in BR_n^V(p_{-n})$.

Now, if $V^n(p_{-n})$ is not in the interior of any cone, then it belongs to the intersection (of the boundary) of some cones: assume therefore $V^n(p_{-n}) \in \bigcap_{i=1}^{k_n} N_{P_n}(\tilde{v}_i)$. We want to show that for $\lambda > 0$ sufficiently large $r_n(p_n + \lambda V^n(p_{-n})) = \sum_{i=1}^{k_n} \alpha_i \tilde{v}_i$, where $\alpha_i \geq 0$, $\sum_{i=1}^{k_n} \alpha_i = 1$, which implies that $r_n(p_n + \lambda V^n(p_{-n})) \in BR_n^V(p_{-n})$.

The following two properties hold and will be useful for the remainder of the proof and can be easily checked:

(1) $\langle V^n(p_{-n}), \tilde{v}_i - \tilde{v}_j\rangle = 0$, $i, j \in \{1, ..., k_n\}$.
(2) If $V^n(p_{-n}) \notin N_{P_n}(v)$, then for all $\tilde{v} \in \{\tilde{v}_1, ..., \tilde{v}_{k_n}\}$, it implies that $\langle V^n(p_{-n}), \tilde{v} - v\rangle > 0$.

Assume by contradiction there exist $(\lambda_k)_{k\in\mathbb{N}}$ with $\lambda_k \uparrow +\infty$, $z_n^k := p_n + \lambda_k V^n(p_{-n})$ and $\{v_s^k\}_s \subset V(P_n)$ with $V^n(p_{-n}) \notin N_{P_n}(v_s^k), \forall s$, such that $r_n(z_n^k) = \sum_i \alpha_i^k \tilde{v}_i^k + \sum_s \beta_s^k v_s^k$, with $\alpha_i^k \geq 0, \beta_s^k > 0$ and $\sum_i \alpha_i^k + \sum_s \beta_s^k = 1$. Without loss of generality, we can assume $\tilde{v}_i^k = \tilde{v}_i, v_s^k = v_s$, $\forall k$. Let $\tilde{v} \in \{\tilde{v}_1, ..., \tilde{v}_{k_n}\}$. Then property (1) implies $\langle z_n^k - r_n(z_n^k), \tilde{v} - r_n(z_n^k)\rangle = \langle p_n - r_n(z_n^k), \tilde{v} - r_n(z_n^k)\rangle + \lambda_k \sum_s \beta_s^k \langle V^n(p_{-n}), \tilde{v} - v_s^k\rangle$. Since $\langle z_n^k - r_n(z_n^k), \tilde{v} - r_n(z_n^k)\rangle \leq 0$, property (2) implies that $\sum_s \beta_s^k \to 0$, as $k \to +\infty$. Therefore, $r_n(z_n^k) \to \sum_i \alpha_i \tilde{v}_i$, as $k \to +\infty$. Let $c(\alpha) := \sum_i \alpha_i \tilde{v}_i$.

We show now that for k sufficiently large, $\langle p_n + \lambda_k V^n(p_{-n}) - c(\alpha), p' - c(\alpha)\rangle \leq 0, \forall p' \in P_n$. This implies that for k sufficiently large $\sum_s \beta_s^k = 0$, which is a contradiction and finishes the proof of the claim.

Let $\bar{v}_k \in \arg\max_{\tilde{p}\in\tilde{P}} \langle p_n + \lambda_k V^n(p_{-n}) - c(\alpha), \tilde{p} - c(\alpha)\rangle$. Without loss of generality assume $\bar{v}_k = \bar{v}, \forall k$. Notice that $\langle V^n(p_{-n}), \bar{v} - c(\alpha)\rangle \leq 0$, since by assumption $V^n(p_{-n})$ is in the cones $N_{P_n}(\tilde{v}_i)$.

Assume first by contradiction that $\langle p_n - c(\alpha), \bar{v} - c(\alpha)\rangle > 0$. If $\langle V^n(p_{-n}), \bar{v} - c(\alpha)\rangle = 0$, then we have that $\langle V^n(p_{-n}), \bar{v} - \tilde{v}_i\rangle = 0$, $\forall \tilde{v}_i$. Now it follows that $\langle V^n(p_{-n}), \bar{v} - r_n(z_n^k)\rangle = \langle V^n(p_{-n}), \bar{v} - \tilde{v}_i + \tilde{v}_i - r_n(z_n^k)\rangle = \sum_s \beta_s^k \langle V^n(p_{-n}), \tilde{v}_i - v_s\rangle > 0$, by property (2). Therefore, for k sufficiently large, $\langle p_n + \lambda_k V^n(p_{-n}) - r_n(z_n^k), \bar{v} - r_n(z_n^k)\rangle > 0$, which violates the definition of the nearest point retraction. Therefore we have that $\langle V^n(p_{-n}), \bar{v} - c(\alpha)\rangle < 0$. But then for k sufficiently large, it follows that $\langle p_n + \lambda_k V^n(p_{-n}) - c(\alpha), \bar{v} - c(\alpha)\rangle < 0$. Hence, for k sufficiently large, $r_n(z_n^k) = c(\alpha)$, which is a contradiction.

It follows that $\langle p_n - c(\alpha), \bar{v} - c(\alpha)\rangle \leq 0$. But then, since $\langle V^n(p_{-n}), \bar{v} - c(\alpha)\rangle \leq 0$, it follows that $\langle p_n + \lambda_k V^n(p_{-n}) - c(\alpha), \bar{v} - c(\alpha)\rangle \leq 0$. This implies that $r_n(z_n^k) = c(\alpha)$. Contradiction. This finishes the proof of claim 7.12.

We claim that for large enough $\lambda > 0$ the map $g^\lambda : P \to P$ defined by $g^\lambda := \times_n(r_n \circ w_n^\lambda)$ with $w_n^\lambda(p) = p_n + \lambda V^n(p_{-n})$ satisfies $\text{Graph}(g^\lambda) \subset O$, where O is the open neighborhood of the graph of the best-reply BR^V defined in the beginning of subsection 3.2.2.

Suppose by contradiction the claim is not true. Then there exists a sequence $\lambda_k \to +\infty$ as $k \to \infty$ such that $g^k := g^{\lambda_k}$ satisfies $\text{Graph}(g^k) \cap O^c \neq \emptyset$ for all k. Since P is compact and g^k continuous, $(\text{Graph}(g^k))_{k\in\mathbb{N}}$ is a sequence of non-empty compact subsets of $P \times P$. This implies we can extract a convergent subsequence (in the Hausdorff-distance) of $(\text{Graph}(g^k))_{k\in\mathbb{N}}$ to a nonempty compact subset of $P \times P$. Assume without loss of generality that $\text{Graph}(g^k)$ converges to a nonempty compact set F in the Hausdorff-distance. It follows that $F \cap O^c \neq \emptyset$. Let $z \in F \cap O^c$. Consider B_z an open neighborhood of z that does not intersect $\text{Graph}(BR^V)$. Therefore we have that $\text{Graph}(g^k) \cap B_z \neq \emptyset$ for sufficiently large k.[11] This implies that there exists an open set U in P such that, for k large enough, $\forall p \in U, (p, g^k(p)) \in B_z$.

Fix $p \in U$. By claim 7.12, it follows that for large enough k, $r_n(p_n + \lambda_k V^n(p_{-n})) \in BR_n^V(p_{-n}), \forall n \in N$. Therefore, for k large enough, we have that $g^k(p) \in BR^V(p)$. This implies that for k sufficiently large $(p, g^k(p)) \in (B_z)^c$. Contradiction.

Hence there exists $\lambda_0 > 0$ such that $\forall\lambda \geq \lambda_0$ we have $\text{Graph}(g^\lambda) \subset O$. Now define the homotopy $H : [0, 1] \times \Sigma \to \Sigma$ such that $H(t, \cdot) = g^{1+t(\lambda-1)}(\cdot)$. Notice that the games denoted by $V^{1+t(\lambda-1)}$ – whose payoffs are given by $[1 + t(\lambda - 1)]V_n$, for each player n – all have the same equilibria, which implies that their associated GPS-maps $g^{1+t(\lambda-1)}$ all have the same fixed-points. Therefore the homotopy H preserves fixed-points. This implies that the indices of a component of equilibria C under g^1 and g^λ are identical, by the homotopy property of the index (Theorem 5.15, Chapter VII in Dold (2012)). Since $\text{Graph}(g^\lambda)$ is contained in the neighborhood O of $\text{Graph}(BR^V)$, this implies that the index of C under g^1 - the GPS-map - is identical to $\text{Ind}_{BR^V}(C)$, which concludes the proof. □

Proof of Theorem 4.1. Lemmata 7.11 and 7.10 prove Theorem 4.1 for standard strategic-form games. Now the result follows from Propositions 3.4, 3.12, 3.14. □

7.4. **Proofs of Section 5.** We introduce a few definitions and prove a slightly more general version of Theorem 5.1. This version will also be used in proving results of section 6.

Recall that we defined in subsection 3.2.2 the best-reply index of a component of equilibrium D by choosing a suitable neighborhood U of that component that contained no equilibrium in its boundary as well as a continuous function $f : P \to P$ whose graph is "sufficiently close" to the graph of the best-reply correspondence. We can extend the definition of index of a component to the index of a neighborhood – straightforwardly by repeating the reasoning of subsection 3.2.2 – to any open neighborhood in P that does not contain any equilibrium in its boundary. An open neighborhood in P with this property will be called *suitable*. Let U be suitable. We define *the local index of the best-reply at an open neighborhood U* as the fixed point index of $f|_U : U \to P$.

[11]This follows from the characterization of the Hausdorff limit F as the closed limit of the sequence $\text{Graph}(g^k)$. See Aliprantis and Border (2006).

Theorem 7.13. *Let $V = (N, (P_n)_{n\in N}, (V_n)_{n\in N})$ and $V^{'} = (N, (P_n')_{n\in N}, (V_n^{'})_{n\in N})$ be two strategic-form games such that for each n there exists an affine function $p_n : P_n \to P_n^{'}$ with a right-inverse $i_n : P_n^{'} \to P_n$ ($p_n \circ i_n = id_{P_n'}$) such that $V_n' \circ p = V_n$, where $p = \times_n p_n, i = \times_n i_n$. Let U be suitable for game V. Then $U' = p(U)$ is suitable for V' and the local index of the best-reply of V at U equals the local index of the best-reply of $V^{'}$ at $U^{'}$.*

Proof. Firstly, if σ is an equilibrium of V, then $p(\sigma)$ is an equilibrium of $V^{'}$, so if U contains no equilibria in its boundary then neither does U'.[12] Let now W be an open neighborhood of the Graph(BR^V) of the best-reply of V such that the local index of the best-reply at U can be calculated from any continuous function $h : P \to P$ with Graph(h) $\subset W$ from the fixed point index of $h|_U$.[13] Consider now an open neighborhood $W^{'}$ of Graph($BR^{V^{'}}$) such that for each $(\sigma', \tau') \in W^{'}$, $(p \times p)^{-1}(\sigma', \tau') \subset W$. By the definition of the local index of the best-reply, there exists a function $h^{'} : P^{'} \to P^{'}$ with Graph($h^{'}$) $\subset W^{'}$ such that the fixed point index of $h^{'}|_{U'}$ is the local index of the best-reply of $V^{'}$ at $U^{'}$. By construction, we have that Graph($i \circ h^{'} \circ p$) $\subset W$. This implies that the fixed point index of $(i \circ h^{'} \circ p)|_U$ equals the local index of the best-reply of V at U. Let $h := i \circ h^{'} \circ p$. Because of the commutativity property of the index in Theorem 5.16 in Chapter VII of Dold (2012), we have that h and $h^{'}$ have homeomorphic sets of fixed points and their indices agree: indeed, defining $h_0 = h^{'} \circ p$ we have that $i \circ h_0 = h$ and $h_0 \circ i = h^{'}$. This implies that the index of $h|_U$ equals the index of $h'|_{U'}$, which concludes the result. □

Corollary 7.14 follows directly from Theorem 7.13 and implies the statement of Theorem 5.1 by setting $p = \Pi$, where $\Pi = \times_n \Pi_n$ and Π_n is the identification mapping defined in subsection 2. Since each Π_n is a surjective affine mapping, it has a right inverse and Theorem 5.1 then follows.

Corollary 7.14. *If C is an equilibrium component of game V then $p(C) =: C^{'}$ is an equilibrium component of $V^{'}$ and the best-reply indices of C and $C^{'}$ are the same.*

Proof of Corollary 5.2. Follows from direct application of Theorem 4.1 and Theorem 5.1. □

7.5. **Proofs of Section 6.2.**

Proof of Theorem 6.12. We start by proving (1). Recall the definition of map $q_n : \Sigma_n \to C_n$: $q_n(\sigma_n) = p_n$, where $p_n(i) = \sum_{s\in s_n(i)} \sigma_n(s), \forall i \in L_n$. Consider $\sigma_n, \sigma_n' \in q_n^{-1}(p_n)$. Then by definition $p_n(i) = \sum_{s\in s_n(i)} \sigma_n(s) = \sum_{s\in s_n(i)} \sigma_n'(s), \forall i \in L_n$. This in turn implies that σ_n, σ_n' are equivalent, in the sense that they induce the same distribution over terminal nodes of the game-tree of G. This implies that for each player $j \in N$ and profile of pure strategies $s_{-n} \in S_{-n}$, $\tilde{G}_j(\sigma_n, s_{-n}) = \tilde{G}_j(\sigma_n', s_{-n})$, which implies that $\sigma_n, \sigma_n' \in [\sigma_n]$, where $[\sigma_n]$ is the equivalence class defined in section 2. We define $\bar{q}_n : C_n \to P_n$ by $\bar{q}_n(p_n) := [\sigma_n]$ and $\bar{q} = \times_n \bar{q}_n$, where any two $\sigma_n, \sigma_n' \in q_n^{-1}(p_n)$ satisfy $\sigma_n, \sigma_n' \in [\sigma_n]$. Recall that $[\sigma_n]$ can be taken to be a point in P_n, the polytope obtained from the partition mapping Π_n. Surjectivity and the affine property follow directly from the definition of $\bar{q}_n$. Let $\bar{q} = \times_n \bar{q}_n$. Notice that $\Pi = \bar{q} \circ q$, which implies that $V_n \circ \bar{q} \circ q = V_n \circ \Pi = \tilde{G}_n$. From the definition of q, it follows that $V_n' \circ q = \tilde{G}_n$. Since q_n is affine and

[12] This follows from the fact that the map p_n is an affine and surjective mapping, so it is an open mapping. This plus the Closed Map Lemma implies that p is an open and closed map, which implies the statement.

[13] Recall from subsection 3.2.2 that if the neighborhood W is well chosen then the local index of the best-reply at U can be calculated from any function $h : P \to P$ with Graph(h) $\subset W$.

surjective it has a right-inverse e_n and $e = \times_n e_n$ is a (multiaffine) right inverse to q. This implies then that $V_n \circ \bar{q} = V'_n$.

Claim (2) is a consequence of Corollary 7.14 and Theorem 4.1. $\square$

We recall some facts from section 7.3 for the proof of Theorem 6.12.

Proof of Theorem 6.14. We denote by $G \bigoplus g$, where $g = (g_n)_{n \in N}$ and $g_n \in \mathbb{R}^{L_n}$, the extensive-form game where, for each player n, the payoffs over terminal nodes are given by $(G_n(z) + g_n(l_n(z)))$, if $l_n(z) \neq \emptyset$ and $G_n(z)$, if $l_n(z) = \emptyset$. Let $\mathcal{E}_G = \{(g,p) \in \Pi_n \mathbb{R}^{L_n} \times \Pi_n C^{\epsilon}_n |\ p$ is an equilibrium of the game $G \bigoplus g\}$. Define now $\theta_G : \mathcal{E}_G \to \Pi_{n \in N} \mathbb{R}^{L_n}$ by $(\theta_G)_n(g,p) = (p_n(i) + \nu_n(i, p_{-n}) + g_n(i))_{i \in L_n}$. Using the same reasoning as in Theorem 7.10 we can show that θ_G is a homeomorphism and $\theta_G^{-1}(g) = (f(g), r(g))$, where f is the permuted GPS-map Φ_G defined before the statement of the Theorem 6.14 . Define $\text{proj}_g : \mathcal{E}_G \to \Pi_n \mathbb{R}^{L_i}$ as $\text{proj}_g(g,p) = g$. Then it implies that $\text{proj}_g \circ \theta_G^{-1}(g) = f(g)$.

Let now $\Theta_{GW} : \mathcal{E}^{GW} \to \mathcal{G}$ be the homeomorphism presented in Theorem 5.2 in Govindan and Wilson (2002). As in Kohlberg and Mertens (1986), we can reparametrize the graph of equilibria $\mathcal{E}^{GW}$ writing an element $(G,p) \in \mathcal{E}^{GW}$ as $(\tilde{G}, g, p)$, where G is uniquely written as $G = \tilde{G} \bigoplus g$, with $\tilde{G}_n(z) := G_n(z) - \mathbb{E}[G|Z_n(l_n(z))]$ and $g_n(z) := \mathbb{E}[G|Z_n(l_n(z))]$, if $l_n(z) \neq \emptyset$ and $\tilde{G}_n(z) = G_n(z)$, if otherwise. Notice that such a vector g of the decomposition can be assumed to be in $\Pi_n \mathbb{R}^{L_n}$ since, for $z, z' \in Z$ with $l_n(z) = i = l_n(z')$ we have that $g_n(z) = g_n(z')$ and if $z \in Z$ is such that $l_n(z) = \emptyset$ we have that $g_n(z) = 0$. Then Θ_{GW} is defined by $\Theta_{GW}(\tilde{G}, g, p) = (\tilde{G}, t)$, where $\tilde{G} := (\tilde{G}_n)_{n \in N}$, $\tilde{G}_n := (\tilde{G}_n(z))_{z \in Z}$ and $t \in \Pi_n \mathbb{R}^{L_n}$ with $t_n(i) = p_n(i) + \nu_n(i, p_{-n}), i \in L_n$. The payoff over terminal nodes represented in the vector $(\tilde{G}, t)$ is given by $\tilde{G}_n(z) + t_n(l_n(z))$. Notice that Θ_{GW} is the identity in the first coordinate, similarly to what happened with the homeomorphism θ in Theorem 7.3.

For the fixed game $G \in \mathcal{G}$ consider now its decomposition $(\tilde{G}, g)$. Define the graph $\mathcal{E}_{\tilde{G}} := \{(g', p) \in \Pi_n \mathbb{R}^{L_n} \times \Pi_n C^{\epsilon}_n |\ p$ is an equilibrium of $\tilde{G} \bigoplus g'\}$ and $\theta' : \mathcal{E}_{\tilde{G}} \to \Pi_n \mathbb{R}^{L_n}$ defined by $\theta'(g', p) = t$, where t satisfies $\Theta_{GW}(\tilde{G}, g', p) = (\tilde{G}, t)$. Therefore, θ' is a homeomorphism. Let $\text{proj}' : \mathcal{E}_{\tilde{G}} \to \Pi_n \mathbb{R}^{L_n}$ be the projection over the first coordinate. We can define the local degree of an equilibrium component C of game G as the local degree of the projection $\text{proj}'|_V$ restricted to an open neighborhood V in $\mathcal{E}_{\tilde{G}}$ of $\{g\} \times D$ that contains no other pair (g,p) in its boundary. This local degree, by the same argument as in the proof of Lemma 7.10, agrees with the degree of D calculated from $\text{proj}_{\mathcal{G}} : \mathcal{E}^{GW} \to \mathcal{G}$.

Note now that $\theta_G(\{0\} \times D) = \theta'(\{g\} \times D)$. Letting $f' := \text{proj}' \circ \theta'^{-1}$, we have therefore that $f = f' - g$. Therefore f and f' have the same local degrees. Since f is the displacement map of the permuted $GPS-$map, it follows that the local degree of f and the index of the GPS map are equal, which implies $\text{ind}(D, \Phi_G)) = \deg_G^{GW}(D)$. $\square$

Proof of Lemma 6.16. Let P^{ϵ}_n denote a polytope in the interior of simplex Σ_n such that $d(P^{\epsilon}_n, \Sigma_n) \leq \epsilon$. Let $P^{\epsilon} := \times_n P^{\epsilon}_n$ and $BR^{G|_{P^{\epsilon}}}$ be the best-reply correspondence of game $G|_{P^{\epsilon}}$. Let $r^{\epsilon}_n : \Sigma_n \to P^{\epsilon}_n$ be the nearest-point retraction and $r^{\epsilon} := \times_n r^{\epsilon}_n$. Let $i_n : P^{\epsilon}_n \to \Sigma_n$ be the inclusion map and $i := \times_n i_n$. Define the correspondence $\Gamma_{\epsilon} : \Sigma \rightrightarrows \Sigma$ by $\Gamma_{\epsilon}(\sigma) = (i \circ BR^{G|_{P^{\epsilon}}} \circ r^{\epsilon})(\sigma)$. Notice that $\text{Graph}(\Gamma_{\epsilon})$ converges (in the Hausdorff-distance) to $\text{Graph}(BR^G)$ as $\epsilon \to 0$. Let W be an open neighborhood of the $\text{Graph}(BR^G)$ that does not intersect $\{(\sigma, \sigma) \in \Sigma \times \Sigma | \sigma\ \text{cl}(U) \setminus U\}$ and according to which the local index of the best-reply BR^G at U can be calculated from any continuous map $h : \Sigma \to \Sigma$ with $\text{Graph}(h) \subset W$. Then, for $\epsilon > 0$ sufficiently small we have that $\text{Graph}(\Gamma_\epsilon) \subset W$. Let $U^\epsilon := U \cap P^\epsilon$. Taking further $\epsilon > 0$ sufficiently small, then U^ϵ contains no equilibria of $G|_{P^\epsilon}$ in its boundary (in P^ϵ). Then there exists a continuous function $h^\epsilon : P^\epsilon \to P^\epsilon$ such that the fixed point index of $h^\epsilon|_{U^\epsilon}$ is well defined and equals the local index of the best-reply of $G|_{P^\epsilon}$ at U^ϵ (see McLennan (1989)). Moreover, we can assume $\text{Graph}(i \circ h^\epsilon \circ r^\epsilon) \subset W$. This implies that the fixed point index of $(i \circ h^\epsilon \circ r^\epsilon)|_U$ is equal to the local index of the best-reply of G at U. Finally, the commutativity property of the fixed point index (Theorem 5.16 in Chapter VII of Dold (2012)) shows that the fixed point index of $(i \circ h^\epsilon \circ r^\epsilon)|_U$ equals the fixed point index of $h^\epsilon|_{U^\epsilon}$, which is the local index of the best-reply of $G|_{P^\epsilon}$ at U^ϵ. □

Proof of Proposition 6.15. Let $\phi : \mathcal{G} \times C^\epsilon \to C^\epsilon$ be defined as $\phi(G, p) = \Phi_G(p)$. The map ϕ is continuous and is therefore a *Nash-map*, that is, a continuous map such that for each $G \in \mathcal{G}$, $\phi(G, \cdot)$ has as fixed points the equilibria of game G.

Using the representation mapping $R : \Pi_n A^\circ(C^\epsilon) \to \mathbb{R}^{N|Z|}$ from Corollary 6.10, let $\tilde{\phi} : A^\circ(C^\epsilon) \times C^\epsilon \to C^\epsilon$ be defined by $\tilde{\phi} = \phi \circ (R \times id_{C^\epsilon})$. Then $\tilde{\phi}$ is also a Nash-map for the strategic-form games with payoff functions $V_n \in A^\circ(C^\epsilon)$, for each player $n \in N$. Notice that $\phi(G, \cdot) = \tilde{\phi}(V, \cdot)$, where $R(G) = V$, so the fixed points and the indices assigned to these fixed points are the same according to the two Nash-maps ϕ and $\tilde{\phi}$.

Now recall from proof of Theorem 3.1 that the function $T : \Pi_n A^\circ(C^\epsilon) \to \mathbb{R}^{NM}$ is a linear isomorphism from the multiaffine functions over C^ϵ to $\mathbb{R}^{NM}$, where M is appropriately defined. Let P^s_n be the standard polytope resulting from an embedding e_n of C^ϵ_n in $\mathbb{R}^{d_n}$, where the dimension of C^ϵ_n is $d_n - 1$. Let $V^s = (N, (P^s_n)_{n\in N}, (V^s_n)_{n\in N})$ be the standard strategic-form game obtained from $V = (N, (C^\epsilon_n)_{n\in N}, (V_n)_{n\in N})$. Now let $\bar{\phi} : \mathbb{R}^{NM} \times P^s \to P^s$ be defined by $\bar{\phi} = e \circ \tilde{\phi} \circ (T^{-1} \times e^{-1})$. Notice that $\bar{\phi}(V^s, \cdot) = e \circ \tilde{\phi}(V, \cdot) \circ e^{-1}$ which implies, by the comutativity property of the index, that the fixed point sets of $\tilde{\phi}$ and $\bar{\phi}$ are homeomorphic and their indices agree. Also, by construction $\bar{\phi} : \mathbb{R}^{NM} \times P^s \to P^s$ is a Nash-map.

It follows that: $\deg_V(D) = \deg_{V^s}(e(D)) = \text{ind}_{GPS-V^s}(e(D)) = \text{ind}(e(D), \bar{\phi}(V^s, \cdot)) = \text{ind}(D, \phi(V, \cdot)) = \text{ind}(D, \Phi_G) = \deg^{GW}_G(D)$, where the first inequality follows from Corollary 7.4, the second from Lemma 7.10, the third from Lemma 7.6, the fourth from the observation in the previous paragraph, the fifth from definition and the sixth from Theorem 6.14. □

Proof of Theorem 6.19. From Govindan and Wilson (2005), it follows that the best-reply index $\text{ind}_{BR^G}(C)$ equals $\deg_G(C)$. Now, Corollary 6.18 gives us that $\text{ind}_{BR^G}(C)$ equals the local index of the best-reply BR^V at U', where V is the strategic-form of the extensive-form game G. By the additivity property of the index (see Dold (2012), 5.6, Chapter VII), since U' does not contain any equilibrium in its boundary, the local index of the best-reply of V at U' equals the sum of the best-reply indices of the equilibrium components $C' \subset U'$. Since the $\text{ind}_{BR^V}(C') = \deg_V(C')$ by Proposition 4.1 and $\deg_V(C') = \deg^{GW}_G(C')$ by Proposition 6.15, the result follows. □

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