# INFORMATION SPILLOVER IN A BAYESIAN REPEATED GAME

LUCAS PAHL

Abstract. This paper considers an infinitely repeated three-player Bayesian game with lack of information on two sides, in which an informed player plays two zero-sum games simultaneously at each stage against two uninformed players. This is a generalization of Aumann, Maschler and Stearns (1995) two-player zero-sum one sided incomplete information model. Under a correlated prior, the informed player faces the problem of how to optimally disclose information among two uninformed players in order to maximize his long term average payoffs. The objective is to understand the adverse effects of "information spillover" from one game to the other in the equilibrium payoff set of the informed player. We provide conditions under which the informed player can fully overcome such adverse effects, and show that in some cases the adverse effects are unsurmountable and severe.

## 1. Introduction

In their seminal work, Aumann, Maschler and Stearns (1995) analyzed an undiscounted infinitely repeated two-person zero-sum game with lack of information on one side: one player (the informed) knows the stage game being played whereas the other (the uninformed) does not know and cannot observe payoffs, only actions. They showed that this game has a value and constructed optimal strategies for the players. Matters are more complicated if the informed player were to play against *more than one* uninformed player, as it would be the case of a military power (e.g., USA) negotiating with two different countries (e.g., Russia and Iran).[1] By observing what the informed player plays against some other uninformed player, an uninformed player can make inferences about the game he plays against the informed player. As a consequence, it may not be optimal for the informed player to play his unilaterally optimal strategy against some of the uninformed players. Put differently, the *information spillover* among the games played between the informed player and the uninformed players adds layers of complexity to the analysis. Is it still possible to ensure the existence of an equilibrium in general? Is it possible for the informed player to overcome the negative effects of information spillover?

This paper answers such questions. We consider a three-player undiscounted infinitely repeated game in which one of the players is informed of the two zero-sum stage games that he plays against each of the other two (uninformed) players. Each uninformed player only knows the prior probability distribution over the finite set of pairs of zero-sum finite-action stage games, and during the play of the game observes the profiles of actions (but not the payoffs). The informed player collects the sum of payoffs from the two component games. Observe that the resulting game is not zero-sum (and it is a Bayesian game), so it is not clear a priori

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[1]The USA may want to conceal from Russia the exact size of its arsenal, and at the same time may want to reveal it to Iran to leverage its bargaining position. More examples in this line can be found in Aumann, Maschler and Stearns (1995).

that a (uniform) equilibrium necessarily exists. We nevertheless establish the existence of an equilibrium, extending the ideas from Sorin (1983) and Simon, Spież and Toruńczyk (1995) to our setting. The specific model we analyse, with one informed player playing two zero-sum games against two uninformed players, allows us to obtain natural lower and upper bounds for the equilibrium payoffs of the informed player[2]. The existence result shows that the lower bound can always be obtained in equilibrium. The next question is whether it is possible for the informed player to obtain the upper bound in equilibrium. Obviously, if the two games are drawn independently, then we have two separate games played in parallel and there is no adverse effect coming from information spillover. So, in such case, a uniform equilibrium obviously exists, the upper bound equals the lower bound and the informed player can obtain the upper bound by simply playing optimally in each game separately. Hence, the adverse effects of information spillover can only take effect when the games are not drawn independently. Perhaps surprisingly, we establish that the informed player can obtain the upper bound in equilibrium when the component zero-sum games are *locally nonrevealing*, independently of how correlated the component games are.[3] We actually establish more: when the component games are locally nonrevealing, the entire interval from the lower to the upper bound can be obtained in equilibrium, so we have an equilibrium payoff characterization. Finally, we derive a necessary condition for the upper bound to be obtained in equilibrium, and use it to construct an example of a game where the lower bound is not equal to the upper bound and it is the only achievable equilibrium payoff for the informed player.

1.1. **Related Literature.** To the best of our knowledge, the model analyzed here is new and the immediately related papers are the ones cited above. We highlight here a few additional papers in the discounted and undiscounted Bayesian repeated games with perfect monitoring literature that have technical and thematic similarities to this project. A significant part of the literature in undiscounted Bayesian repeated games with perfect monitoring analyses models under the assumption of "known own payoffs" (see Forges (1992)). This is a reasonable assumption in several applications and allows for equilibrium-payoff characterizations which are especially tractable (see Shalev (1988), Koren (1992)). Under this assumption, Forges and Salomon (2015) provide a simple characterization of Nash equilibrium payoffs in undiscounted Bayesian repeated games of two players.[4] This characterization is used to show that in a class of public good games, uniform equilibria might not exist. More closely related to our paper in terms of the information environment is Forges, Horst and Salomon (2016). In this paper, among other results, cooperative solutions of one-shot Bayesian games with two players and exactly one informed player are related to noncooperative solutions of two-player repeated Bayesian games with exactly one informed player. More specifically, under the assumption of existence of uniform punishment strategies for the uninformed player, the *joint-plan* equilibrium payoffs of the repeated game equal the set of cooperative solutions of the one-shot Bayesian game. This

[2]The informed player can play the optimal strategy of the "sum of the two zero-sum games" he plays, guaranteeing the referred lower bound. Analogously, each uninformed player has the optimal strategy of his own zero-sum game, which guarantees the informed player cannot obtain more than the sum of the values of each zero-sum game, which is the referred upper bound.

[3]Informally, a component game is *locally nonrevealing* if any optimal strategy of the informed player induces at least one posterior belief with full support.

[4]An additional assumption needed for the characterization is the existence of "uniform punishment strategies" for the players in the Bayesian stage game, that is, strategies that allow a player to be punished in his ex-post individually rational level.

*folk theorem* is not however an equilibrium payoff characterization, since it is known from Hart (1985) that joint-plans cannot account for the whole of equilibrium payoffs in general.[5]

Although concerned with discounted Bayesian repeated games, the study of "belief-free equilibria" in Hörner, Lovo and Tomala (2011) has technical similarities to the study of uniform equilibria in undiscounted games. As noted by the authors, the set characterizing belief-free equilibrium payoffs (called $V^*$) appears in the literature in undiscounted Bayesian repeated games multiple times (see Forges (1992)). The conditions defining $V^*$ reappear in our paper, specified to our information environment: they are the sufficient conditions for both our version of joint-plans to induce uniform equilibrium payoffs.[6] Hörner, Lovo and Tomala (2011) consider different assumptions on payoffs and information environment in order to provide sufficient conditions under which $V^*$ would be nonempty. None of the cases considered can be immediately applied here in order to obtain existence or our characterization of equilibrium payoffs in our model.

One additional reference in the discounted dynamic games literature is Huangfu and Liu (2015). Although considering a significantly different model from ours, this paper motivated our work by considering information spillovers between different markets with asymmetric information. In that paper, a seller holds private information about the quality of goods he sells in two different markets and buyers learn about the seller's private information from observing past trading outcomes not only in the market on which they directly participate, but also from observing the outcomes of the other market. The authors show, under certain assumptions on the correlation of qualities between goods in different markets, that information spillover can mitigate adverse selection.

1.2. **Organization.** The remaining of the paper is organized as follows. The model is presented in Section 2. Section 3 establishes existence of equilibrium; Section 4 presents an equilibrium payoff characterization; Section 5 constructs an example with only one equilibrium payoff (the lower bound), where the upper bound cannot be attained. Proofs of lemmas and technical results are left to the Appendix. Additional results can be found in an online appendix.[7]

## 2. Model

2.1. **Notation.** Given a finite set $K$, $\Delta(K)$ is the set of probability distributions over $K$; for $X \subset \mathbb{R}^m$, the interior of $X$ will be denoted by $\text{int}X$, its boundary by $\partial X$, and its convex closure by $\text{co}(X)$. For $p \in \Delta(K_A \times K_B)$, $p_A$ (resp. $p_B$) denotes its marginal on $K_A$ (resp. $K_B$), and $\text{supp}(p)$ its support. We denote a product distribution on $K_A \times K_B$ by $p_A \bigotimes p_B$, and use $\Delta(K_A) \bigotimes \Delta(K_B)$ to denote the set of all such distributions.

A *3-player infinitely repeated game with lack of information on two sides* is given by

$$\mathcal{G}(p) = (\{1, 2, 3\}, I_A, I_B, J_A, J_B, K_A, K_B, p, A, B),$$

where:

- $\{1, 2, 3\}$ is the set of players, 1 being the informed and the other two uninformed.

[5]We note that a simple modification of Hart's characterization – in order to include an additional uninformed player – holds for our model.

[6]Differently from Hörner, Lovo and Tomala (2011), here "joint rationality" conditions have no bite, since only one player is informed on payoffs. A version of the incentive compatibility condition appears here as Condition (2) in Lemma 3.6, and the other conditions of the Lemma are simply individual rationality conditions.

[7]https://pahllucas.wixsite.com/lucaspahl.

- $I_i$, $J_i$, $K_i$, $i = A, B$ are finite sets, with $I_A \times I_B$ (resp. $J_A$ and $J_B$) being the set of actions of player 1 (resp. players 2 and 3), and $K_A \times K_B$ being the set of states.
- $p \in \Delta(K_A \times K_B)$ is the prior.[8]
- $A : K_A \to \mathbb{R}^{|I_A| \times |J_A|}$ and $B : K_B \to \mathbb{R}^{|I_B| \times |J_B|}$ are functions that associate to each $k_A \in K_A$ and $k_B \in K_B$ zero-sum payoff matrices $A^{k_A} \in \mathbb{R}^{|I_A| \times |J_A|}$ and $B^{k_B} \in \mathbb{R}^{|I_B| \times |J_B|}$.

The play of the infinitely repeated game is as follows: states $(k_A, k_B)$ are drawn according to distribution $p$ and player 1 knows the draw, whereas players 2 and 3 do not. The stage game payoff for player 1 in case actions $i_A, i_B, j_A, j_B$ realize is $A^{k_A}_{i_A, j_A} + B^{k_B}_{i_B, j_B}$; for player 2, $-A^{k_A}_{i_A, j_A}$ and for player 3, $-B^{k_B}_{i_B, j_B}$. The stage game is infinitely repeated, with actions at each stage being publicly observable. Players are Bayesian, are assumed to have perfect recall and the whole description of the game is common knowledge. If $h_m = (i^1_A, i^1_B, j^1_A, j^1_B, ..., i^{m-1}_A, i^{m-1}_B, j^{m-1}_A, j^{m-1}_B)$ is a history of moves in $\mathcal{G}(p)$ up to stage $m$, $h^A_m := (i^1_A, j^1_A, ..., i^{m-1}_A, j^{m-1}_A)$ denotes the history of moves of player 1 and player 2 up to stage m; $h^B_m$ is defined analogously.

For $t \in \mathbb{N}^*$ (the set of positive integers), $H_t := (I_A \times J_A \times I_B \times J_B)^{t-1}$ is the set of histories before stage $t$ ($H_1$ is the "empty" history, since there is no information to condition on). We also define the set of infinite histories by $H_\infty = \Pi^\infty_{t=1}(I_A \times J_A \times I_B \times J_B)$, an element of $H_\infty$ being a sequence $h_\infty = (i^t_A, i^t_B, j^t_A, j^t_B)_{t \in \mathbb{N}^*}$ of moves by all players at all stages.

2.2. **Strategies.** Pure strategies for player 1 are maps $s^1 : (K_A \times K_B) \times \bigcup_{t \in \mathbb{N}^*} H_t \to I_A \times I_B$; for player 2 (resp. player 3), pure strategies are maps $s^2 : \bigcup_{t \in \mathbb{N}^*} H_t \to J_A$ (resp. $s^3 : \bigcup_{t \in \mathbb{N}^*} H_t \to J_B$). Mixed strategies are distributions over pure strategies. Behavioral strategies for player 1 are defined as the set of maps $\Sigma := \{\sigma : K_A \times K_B \times \bigcup_{t \in \mathbb{N}^*} H_t \to \triangle(I_A \times I_B)\}$. For player 2, the set of maps $\mathcal{T}_2 := \{\tau_A : \bigcup_{t \in \mathbb{N}^*} H_t \to \triangle(J_A)\}$. For player 3, the set of maps $\mathcal{T}_3 := \{\tau_B : \bigcup_{t \in \mathbb{N}^*} H_t \to \triangle(J_B)\}$. By Kuhn's Theorem[9], it is sufficient to consider only behavioral strategies, since they are outcome equivalent to mixed strategies.

2.3. **Payoffs.** For a given profile of strategies $(\sigma, \tau_A, \tau_B)$ and prior $p$ we have a probability distribution $\mathbb{P}_{\sigma, \tau_A, \tau_B, p}$ on the space $\text{supp}(p) \times H_\infty$ (with $\sigma$-field generated by cylinders). Denote by $\mathbb{E}_{\sigma, \tau_A, \tau_B, p}$ the expectation with respect to this distribution, and $\mathbb{E}^{k_A, k_B}_{\sigma, \tau_A, \tau_B, p}$ the conditional expectation with respect to $\kappa_A \times \kappa_B = (k_A, k_B)$, where $\kappa_A \times \kappa_B$ is a random variable representing Nature's randomization. For a fixed duration $T \in \mathbb{N}^*$, define:

- $\alpha^{k_A, k_B}_T(\sigma, \tau_A, \tau_B) := \mathbb{E}^{k_A, k_B}_{\sigma, \tau_A, \tau_B, p}[\frac{1}{T}\sum^T_{t=1}(A^{k_A}_{i^t_A, j^t_A} + B^{k_B}_{i^t_B, j^t_B})]$, for all $(k_A, k_B) \in \text{supp}(p)$, which denotes the expected payoff to player 1 when state is $(k_A, k_B)$ and duration is $T$.
- $\beta^A_T(\sigma, \tau_A, \tau_B) := \mathbb{E}_{\sigma, \tau_A, \tau_B, p}[\frac{1}{T}\sum^T_{t=1}(-A^{\kappa_A}_{i^t_A, j^t_A})]$, which denotes the expected payoff to player 2 when the duration of the game is $T$.
- $\beta^B_T(\sigma, \tau_A, \tau_B) := \mathbb{E}_{\sigma, \tau_A, \tau_B, p}[\frac{1}{T}\sum^T_{t=1}(-B^{\kappa_B}_{i^t_B, j^t_B})]$, which denotes the expected payoff to player 3 when the duration of the game is $T$.

[8]All of the examples in the paper will be such that $p$ is in some proper face of $\Delta(K_A \times K_B)$, so in order to avoid any confusion we will not assume, as it is usual, that $p$ is in the interior of $\Delta(K_A \times K_B)$.

[9]See Appendix D of Sorin (2002).

2.4. **Equilibrium Concept.** A triple of strategies $(\sigma, \tau_A, \tau_B)$ is a uniform equilibrium of $\mathcal{G}(p)$ if both conditions 1 and 2 below are satisfied:

(1) For each $(k_A, k_B) \in \text{supp}(p)$, $(\alpha_T^{k_A,k_B}(\sigma, \tau_A, \tau_B))_{T\geq 1}$ converges as $T$ goes to infinity to some $\alpha^{k_A,k_B}(\sigma, \tau_A, \tau_B)$, $(\beta_T^A(\sigma, \tau_A, \tau_B))_{T\geq 1}$ converges to some $\beta^A(\sigma, \tau_A, \tau_B)$ and $(\beta_T^B(\sigma, \tau_A, \tau_B))_{T\geq 1}$ converges to some $\beta^B(\sigma, \tau_A, \tau_B)$.

(2) For each $\epsilon > 0$, there exists a positive integer $T_0$ such that for all $T \geq T_0$, $(\sigma, \tau_A, \tau_B)$ is an $\epsilon$-Nash equilibrium in the finitely repeated game with $T$ stages, i.e., for each $(k_A, k_B) \in \text{supp}(p)$, $\alpha_T^{k_A,k_B}(\sigma', \tau_A, \tau_B) \leq \alpha_T^{k_A,k_B}(\sigma, \tau_A, \tau_B) + \epsilon, \forall \sigma' \in \Sigma$, $\beta_T^A(\sigma, \tau_A', \tau_B) \leq \beta_T^A(\sigma, \tau_A, \tau_B) + \epsilon, \forall \tau_A' \in \mathcal{T}_2$ and $\beta_T^B(\sigma, \tau_A, \tau_B') \leq \beta_T^B(\sigma, \tau_A, \tau_B) + \epsilon, \forall \tau_B' \in \mathcal{T}_3$.

From now on, "equilibrium" refers to uniform equilibrium.

**Remark 2.1.** If $(\sigma, \tau_A, \tau_B)$ is an equilibrium in $\mathcal{G}(p)$, the associated vector

$$(\alpha(\sigma, \tau_A, \tau_B), \beta^A(\sigma, \tau_A, \tau_B), \beta^B(\sigma, \tau_A, \tau_B)),$$

where $\alpha(\sigma, \tau_A, \tau_B) := (\alpha^{k_A,k_B}(\sigma, \tau_A, \tau_B))_{k_A,k_B}$, is the *vector of payoffs of* $(\sigma, \tau_A, \tau_B)$. Also $\alpha(\sigma, \tau_A, \tau_B) \cdot p$ (where $\cdot$ is the standard scalar product in a Euclidean space) is the *ex-ante equilibrium payoff* of the informed player.

We recall a few concepts of the model of infinitely repeated zero-sum games with lack of information on one side (see Sorin (2002), Ch. 3). Let $K$ be the finite set of states and $(M^k)_{k\in K}$ the collection of zero-sum payoff matrices where $M^k \in \mathbb{R}^{I\times J}$ for each $k \in K$. Denote by $G_M(p)$ the infinitely repeated zero-sum game with lack of information on one side with prior $p \in \Delta(K)$. Define $v_M(p)$ as the *nonrevealing value* of $G_M(p)$, i.e., the value of the one-shot zero-sum game given by the matrix $\bar{M}(p) := \sum_k p^k M^k$. Let $\text{Cav}(v_M)$ be the (pointwise) smallest concave function $g$ from $\Delta(K)$ to $\mathbb{R}$, such that $g(p) \geq v_M(p)$ for all $p \in \Delta(K)$. Alternatively, one can equivalently define $\text{Cav}(v_M)(p) := \{\sum_{i=1}^k \alpha_i v_M(p_i) | \exists k \in \mathbb{N}^*, \forall i \in \{1, ..., k\}, \alpha_i \geq 0, \sum_{i=1}^k \alpha_i p_i = p, \sum_{i=1}^k \alpha_i = 1\}$. Aumann, Maschler and Stearns (1995) proved that the value of $G_M(p)$ exists and equals the *concavification* $\text{Cav}(v_M)$ of $v_M$. They also showed how to construct optimal strategies.

Given the model $\mathcal{G}(p)$, the infinitely repeated zero-sum game with lack of information on one side with prior $p_A$ defined by states $K_A$ and payoff matrices $(A^{k_A})_{k_A\in K_A}$ will be denoted $G_A(p_A)$ – this game is played by players 1 (informed) and 2 (uninformed). Analogously, we define $G_B(p_B)$ as the infinitely repeated zero-sum game with lack of information on one side played between players 1 and 3. The infinitely repeated zero-sum game with lack of information on one side with prior $p \in \Delta(K_A \times K_B)$, where stage payoff matrices are $(C^{k_A,k_B})_{k_A\in K_A, k_B\in K_B}$ given by $C^{k_A,k_B}_{i_A,i_B,j_A,j_B} := A^{k_A}_{i_A,j_A} + B^{k_B}_{i_B,j_B}$ will be denoted $G_{A+B}(p)$.

**Example 2.2.** This example illustrates the new strategic difficulties that arise in the model $\mathcal{G}(p)$ in comparison to the zero-sum model. Two sets $A = \{A^1, A^2\}$ and $B = \{B^1, B^2\}$ of payoff matrices are defined below together with a bistochastic matrix $p^0 \in \Delta(K_A \times K_B)$, where $K_A = \{1, 2\}$ and $K_B = \{1, 2\}$. The set A defines the zero-sum repeated game $G_A(p^0_A)$ and the set $B$ defines $G_B(p^0_B)$.

$$p^0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 4 & 0 & 2 \\ 4 & 0 & -2 \end{bmatrix} \; B^2 = \begin{bmatrix} 0 & 4 & -2 \\ 0 & 4 & 2 \end{bmatrix}$$

In the matrix $p^0$, an entry $p_{ij}$ corresponds to the probability with which Nature chooses $A^i$ and $B^j$. Since the prior assigns perfect correlation between states, there are only two states to consider effectively: states (1,1) and (2,2). Denote by $p$ the probability of state 1 (in both games) and $(1-p)$ the probability of state 2 (in both games). Each row of $A^i$ and $B^j, i, j \in \{1,2\}$ corresponds to a stage-game strategy of the informed player: call the first row "$U$" and the second row "$D$". By computation, we get

$$v_A(p) = p(1-p), \text{ for all } p \in [0,1]$$

$$v_B(p) = \begin{cases} 4p & \text{if } p \in [0,1/4) \\ -4p+2 & \text{if } p \in [1/4,1/2) \\ 4p-2 & \text{if } p \in [1/2,3/4) \\ -4p+4 & \text{if } p \in [3/4,1] \end{cases}$$

These imply that:

$$\mathrm{Cav}(v_A)(p) = v_A(p) = p(1-p), \text{ for all } p \in [0,1]$$

$$\mathrm{Cav}(v_B)(p) = \begin{cases} 4p & \text{if } p \in [0,1/4) \\ 1 & \text{if } p \in [1/4,3/4) \\ -4p+4 & \text{if } p \in [3/4,1] \end{cases}$$

The optimal strategy of the informed player in the repeated zero-sum game $G_B(1/2)$ is defined as follows: in case the state drawn by Nature is 1, the informed player plays "$U$" with probability 1/4 and after that plays at each stage, independently, the optimal strategy of the zero-sum game whose matrix is $\bar{B}(1/4)$; in case the state drawn by Nature is 2, the informed player plays "$U$" with probability 3/4 and, after that, plays the optimal strategy of the zero-sum game whose matrix is $\bar{B}(3/4)$. After observing the realized action of the informed player in the first stage, the uninformed player updates his beliefs about the states to *posteriors* about states 1 and 2: in our example, the uninformed player, after observing "$U$", assigns probability 1/4 to the state being 1. On the other hand, after observing "$D$", the uninformed player assigns probability 3/4 to the state being 1. The strategy just described for the informed player is an example of a *signaling strategy*[10]: the informed player uses his actions to signal information about the underlying state. After the first stage, according to our construction, no more information is signaled and the uninformed player plays the optimal strategy of game $\bar{B}(1/4)$ or $\bar{B}(3/4)$ forever, depending on whether $U$ or $D$ was realized, respectively. Playing the signaling strategy guarantees to the informed player an ex-ante payoff of $(1/2)v_B(1/4) + (1/2)v_B(3/4) = (1/2)\mathrm{Cav}(v_B)(1/4) + (1/2)\mathrm{Cav}(v_B)(3/4) = \mathrm{Cav}(v_B)(1/2) = 1$. Now, in game $G_A(1/2)$ the nonrevealing value function of the informed player is strictly concave, which implies that the optimal strategy in this game is nonrevealing (at any prior): one optimal strategy for the informed player is to play at each stage the optimal

[10]See Lemma 1 in Sorin (1983).

Figure 1. Graphs of Cav($v_A$) and $v_A$

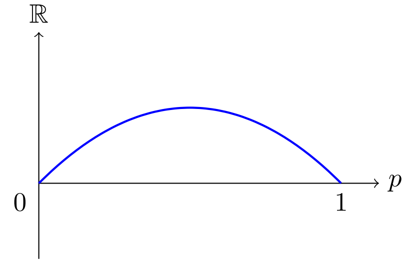
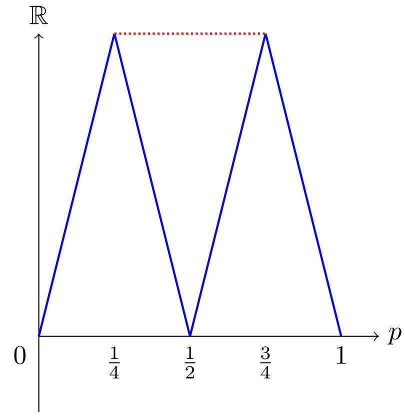



Figure 2. Graphs of Cav($v_B$) and $v_B$




strategy of $\bar{A}(1/2)$ independently forever. If the informed player uses the signaling strategy described in $G_B(1/2)$, because of perfect correlation between $\kappa_A$ and $\kappa_B$, this strategy also induces the same updating on the part of the uninformed player 2, inducing, similarly, posteriors 1/4 and 3/4 in the zero-sum repeated game $G_A(1/2)$. By using that strategy in game $G_B(1/2)$, the informed player in game $G_A(1/2)$ can now only guarantee $(1/2)\mathrm{Cav}(v_A)(1/4) + (1/2)\mathrm{Cav}(v_A)(3/4) = 3/16 < 1/4 = \mathrm{Cav}(v_A)(1/2)$.

In Example 2.2 above, the prior $p^0$ correlates the states in both zero-sum games, generating a "spillover effect" that impairs the ability of the informed player to play the zero-sum games separately. The spillover effect does not exist, however, when $p^0$ is a product distribution $p_A \bigotimes p_B$: in this case, the random variable $\kappa_A$ is independent of the random variable $\kappa_B$. If $\sigma_B$ denotes the signaling strategy of the zero-sum game $G_B(1/2)$ in the example, then $\sigma_B$ is obviously conditioned on the realization of $\kappa_B$, but no signal (e.g., move in first stage) about the value of $\kappa_B$ is informative about the value of $\kappa_A$, since these variables are independent. This implies the informed player can guarantee $\mathrm{Cav}(v_A)(1/2) + \mathrm{Cav}(v_B)(1/2)$ in game $\mathcal{G}(p^0)$. Similarly, the uninformed players can each guarantee $\mathrm{Cav}(v_A)(1/2)$ and $\mathrm{Cav}(v_B)(1/2)$ by playing their optimal strategies in $G_A(1/2)$ and $G_B(1/2)$. This result is indeed general: the informed player can play the optimal strategies of each zero-sum game independently, since no signal in one game is informative about the state of the other. This implies he can guarantee $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$. The uninformed players can play the optimal strategy of their zero-sum games, guaranteeing each $\mathrm{Cav}(v_A)(p_A)$ and $\mathrm{Cav}(v_B)(p_B)$. This gives the following result:

**Proposition 2.3.** *Let $p = p_A \bigotimes p_B$. A uniform equilibrium exists in game $\mathcal{G}(p)$. Moreover, every ex-ante equilibrium payoff of the informed player in $\mathcal{G}(p)$ equals $Cav(v_A)(p_A) + Cav(v_B)(p_B)$.*

Now, given a game $\mathcal{G}(p)$, the number $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$ is an upper bound on the ex-ante equilibrium payoffs of player 1, because each uninformed player can always play the optimal strategy of his repeated zero-sum game, holding the payoffs of the informed player at most $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$.

Let $\bar{h}(p) := v_A(p_A) + v_B(p_B)$. A lower bound on the ex-ante equilibrium payoffs of the informed player is given by the concavification of $\bar{h}$, as the the next proposition establishes.

**Proposition 2.4.** *A lower bound on the ex-ante equilibrium payoffs of the informed player in $\mathcal{G}(p)$ is $Cav(\bar{h})(p)$.*

*Proof.* Consider the infinitely repeated zero-sum game $G_{A+B}(p)$. It is straightforward to check that the nonrevealing value of $G_{A+B}(p)$ is $\bar{h}(p)$. From the results of Aumann, Maschler and Stearns (1995), the value of $G_{A+B}(p)$ is $\mathrm{Cav}(\bar{h})(p)$, so the informed player can guarantee $\mathrm{Cav}(\bar{h})(p)$. □

## 3. Equilibrium Existence

In this section we prove existence of uniform equilibrium.

**Theorem 3.1.** *For each $p \in \Delta(K_A \times K_B)$, there exists a uniform equilibrium in $\mathcal{G}(p)$ where the informed player's ex-ante payoff is $Cav(\bar{h})(p)$.*

A proof of the result is contained in the next subsection, with proofs of auxiliary results left to the Appendix. We look for existence within a particular class of strategies, called *independent and safe joint-plans*. Our definition of independent and safe joint-plans (Definition 3.4) follows closely the definition proposed by Sorin (1983) in the context of two-player nonzero-sum games. We look for uniform equilibria within this class because this is the class where we can leverage some of the results of Simon, Spież and Toruńczyk (1995) of existence of equilibria in nonzero-sum games to our benefit.

Just like in Sorin (1983), Definition 3.4 does not explicitly define the strategies in the repeated game $\mathcal{G}(p)$ but does it implicitly. The idea of a joint-plan is that player 1 uses, during finitely many stages, a signaling strategy to signal his private information over the games $G_A(p_A)$ and $G_B(p_B)$. As we saw in Example 2.2, this signaling procedure can be achieved by player 1 when he conditions his play during the first stages of the game on the realization of the states. After signals are realized, the uninformed players update their prior according to the realized signal. Conditional on the observed signal, players then play according to a deterministic path of play. The idea is that players "agree" on a deterministic path of play after updating according to the information revealed during the first stages of the game. This deterministic path of play is called a "contract" in Definition 3.4. In case player 1 deviates from the prescribed path of play, players 2 and 3 play the optimal strategy of the uninformed player in the zero-sum game $G_{A+B}(p)$ and in case player 2 (player 3) deviates from his prescribed path of play, player 1 plays the optimal strategy of $G_A(p_A)$ ($G_B(p_B)$). Lemma 3.6 provides conditions under which the strategies just described are a uniform equilibrium and Example 3.8 constructs an equilibrium of this kind explicitly, for illustration purposes.

**Remark 3.2.** Before proceeding to the next section, we remark that it might be tempting to use the optimal strategy of the informed player in the zero-sum game $G_{A+B}(p)$ to construct an equilibrium of game $\mathcal{G}(p)$

paying $\mathrm{Cav}(\bar{h})(p)$ to the informed player. A natural idea would be as follows: the informed player would play the optimal strategy of the zero-sum game $G_{A+B}(p)$, which in general would induce finitely many posterior probabilities $(p_s)_{s\in S}$; the uninformed player 2 (respectively player 3), given the realized posterior, say $p_s$, would then play at each stage the optimal strategy of the stage-game $\bar{A}(p_{sA}) = \sum_{k_A\in K_A} p_{sA}^{k_A} A^{k_A}$ (respectively $\bar{B}(p_{sB}) = \sum_{k_B\in K_B} p_{sB}^{k_B} B^{k_B}$). We know already from the theory of repeated zero-sum games with lack of information on one side (see Sorin (2002)) that such strategies for the uninformed players are *not* necessarily optimal or best-replies in each zero-sum game in general (the optimal strategies are "approachability strategies").[11] Therefore, in particular in our model, they are not guaranteed to be optimal or best-replies for the uninformed player. This argument therefore does not imply that the informed player would not have have a profitable deviation from his strategy that would allow him to obtain more than $\mathrm{Cav}(\bar{h})(p)$.

### 3.1. **Equilibrium Construction.**

**Definition 3.3.** Let $h_m^1 := (i_A^t, i_B^t)_{1\le t\le m-1}$. We define $H_n^1 := \bigcup\{h_n^1\}$ and call it the set of individual histories of player 1.

**Definition 3.4.** An *independent joint-plan*[12] in $\mathcal{G}(p)$ is a triple $(S, x, \gamma)$ where:

- (Signals) $S$ is the set of signals, i.e., a subset of $H_n^1$, for some $n \in \mathbb{N}^*$.
- (Signaling Strategy) The vector $x$ is a $|\mathrm{supp}(p)|$-tuple where for each $(k_A, k_B)$ in $\mathrm{supp}(p)$, $x^{k_A,k_B}$ is a probability distribution on $S$.
- (Contracts) $\gamma = (\gamma_A, \gamma_B)$, with $\gamma_i = (\gamma_i^s)_{s\in S}$, and $\gamma_i^s := \sigma_i^s \bigotimes \tau_i^s, \sigma_i^s \in \triangle(I_i)$ and $\tau_i^s \in \triangle(J_i)$, for $i \in \{A, B\}$. We denote by $\gamma_A^s(i_A, j_A)$ the probability of moves $(i_A, j_A)$ and $\gamma_B^s(i_B, j_B)$ the probability of moves $(i_B, j_B)$.

Following Lemma 2 in Sorin (1983), the independent distribution $\gamma_A^s = \sigma_A^s \bigotimes \tau_A^s$ over $I_A \times J_A$ (resp. $\gamma_B^s$) can be induced through the play of a deterministic sequence of moves at each stage by players 1 and 2 (resp. players 1 and 3) with the appropriate frequency. This deterministic path of play is what the contract $\gamma_A^s$ (resp. $\gamma_B^s$) represents.

**Notation.** Let $(S, x, \gamma)$ be an independent joint-plan in $\mathcal{G}(p)$. We define some notation necessary for the statement of Lemma 3.6. The prior $p$ and signaling strategy $x$ define a probability distribution $P$ on $K_A \times K_B \times S$ by letting $P(k_A, k_B, s) := p^{k_A,k_B} x^{k_A,k_B}(s)$. We define the *posterior probability* of $(k_A, k_B) \in \mathrm{supp}(p)$ given the realization of $s \in S$ by $p^{k_A,k_B}(s) := \frac{P(k_A,k_B,s)}{P(s)}$, where $P(s) = \sum_{(k_A,k_B)\in K_A\times K_B} P(k_A, k_B, s)$. This is the probability over states, obtained by Bayes rule, that players 2 and 3 may compute after observing signals. For $s \in S, \ell \in \{A, B\}$, given a posterior probability $p(s) \in \Delta(K_A \times K_B)$, the marginal posterior over $K_\ell$ is $p(s)_\ell \in \Delta(K_\ell)$. Also, for each $s \in S$, $(k_A, k_B) \in \mathrm{supp}(p)$, and $\ell, j \in \{A, B\}$, we define: the expected payoff of player 1 after signal $s$ is $\alpha^{k_A,k_B}(s) := \sum_{i_A,j_A,i_B,j_B} (A_{i_A,j_A}^{k_A} \gamma_A^s(i_A, j_A) + B_{i_B,j_B}^{k_B} \gamma_B^s(i_B, j_B))$; the highest payoff player 1 can obtain after any signal when states are $(k_A, k_B) \in \mathrm{supp}(p)$ is $\alpha^{k_A,k_B} := \max_{t\in S} \alpha^{k_A,k_B}(t)$ and $\alpha = (\alpha^{k_A,k_B})_{(k_A,k_B)\in\mathrm{supp}(p)}$ is their vector; the marginal expected payoff after $s$ of player 1 in game

[11] For example, these strategies do not prevent profitable deviations at signaling stages: the informed player could deviate to a signal in an unobserved way, in case the payoff after playing that signal is higher than for other signals. Approachability strategies, in contrast, also prevent against this type of profitable deviations.

[12] The joint-plan is called "independent" because each contract is defined by a product of strategies of each player.

$\ell \in \{A, B\}$ is $\alpha_\ell^{k_\ell}(s) = \sum_{i_\ell, j_\ell} \ell^{k_\ell}_{i_\ell, j_\ell} \gamma^s(i_\ell, j_\ell)$; we define by $\beta_\ell(s) := \sum_{k_\ell} p^{k_\ell}(s)_\ell \alpha^{k_\ell}(s)$ the negative of the expected payoff of the uninformed player playing game $\ell$ after signal $s$, and by $\beta_\ell := \sum_{s \in S} P(s)\beta_\ell(s)$.

**Definition 3.5.** An independent joint-plan $(S, x, \gamma)$ in $\mathcal{G}(p)$ is called *safe* if for each $s \in S$, $\tau_A^s$ is optimal in the one-shot game with matrix $\bar{A}(p(s)_A) := \sum_{k_A \in K_A} p^{k_A}(s) A^{k_A}$ and $\tau_B^s$ is optimal in the one-shot zero-sum game $\bar{B}(p(s)_B)$.

**Lemma 3.6.** *Let $(S, x, \gamma)$ be an independent joint-plan in $\mathcal{G}(p)$ satisfying:*

1. $\beta_A(s) \leq Cav(v_A)(p(s)_A)$ *and* $\beta_B(s) \leq Cav(v_B)(p(s)_B)$*, for all* $s \in S$.
2. *For all* $(k_A, k_B) \in supp(p)$*,* $s \in S$ *such that* $P(k_A, k_B, s) > 0$*, it implies* $\alpha^{k_A, k_B}(s) = \alpha^{k_A, k_B}$.
3. $\alpha \cdot q \geq \bar{h}(q)$*, for all* $q \in \Delta(supp(p))$.

*Then there exists an equilibrium* $(\sigma, \tau_A, \tau_B) \in \Sigma \times \mathcal{T}_2 \times \mathcal{T}_3$ *in* $\mathcal{G}(p)$ *such that* $\forall (k_A, k_B) \in supp(p)$ *we have* $\alpha^{k_A, k_B}(\sigma, \tau_A, \tau_B) = \alpha^{k_A, k_B}$, $\beta^A(\sigma, \tau_A, \tau_B) = -\beta_A$ *and* $\beta^B(\sigma, \tau_A, \tau_B) = -\beta_B$.

*Proof.* See Appendix □

Given an independent joint-plan $(S, x, \gamma)$, the vector $(\alpha, -\beta_A, -\beta_B) \in \mathbb{R}^{\text{supp}(p)} \times \mathbb{R} \times \mathbb{R}$ will be called the *vector of payoffs* of the equilibrium joint-plan.

Condition (1) in Lemma 3.6 guarantees that players 2 and 3 do not deviate after signaling stages from the deterministic path of play induced by the joint-plan contracts: in case signal $s$ realizes, the contract $\gamma_A^s$ has expected payoff to player 2 given $s$ of $-\beta_A(s)$ and player 1 can punish player 2 in case player 2 deviates from the deterministic path of play given by the contract $\gamma_A^s$ by playing his optimal strategy at $G_A(p(s)_A)$, which guarantees the payoff of player 2 would not be larger than $-\text{Cav}(v_A)(p(s)_A) \leq -\beta_A(s)$. The analogous reasoning holds to prevent a deviation of player 3. Condition (2) prevents undetectable deviations from player 1 at signaling stages: it says that player 1 cannot profit from "lying" about a signal because he is indifferent to the payoffs under any contract that realizes with positive probability, for any pair of states chosen by Nature (see Mertens, Sorin and Zamir (2015), proof of Proposition IX 1.1). Condition (3) implies the existence of strategies for the uninformed players to punish the informed player in case he makes an observable deviation. This is the approachability strategy of the uninformed player in the repeated game $G_{A+B}(p)$.[13]In the Appendix, we show this strategy can indeed be played by players 2 and 3. This will be a simple consequence of the fact that the payoffs for the informed player have a "separable" structure - they are the addition of payoffs obtained in each zero-sum game separately.

**Lemma 3.7.** *There exists an independent and safe joint-plan in* $\mathcal{G}(p^0)$ *satisfying (1), (2) and (3) of Lemma 3.6. Also, if* $(\sigma, \tau_A, \tau_B)$ *is the equilibrium induced by this joint-plan, then* $(\sigma, \tau_A, \tau_B)$ *pays* $Cav(\bar{h})(p)$ *as an ex-ante payoff to the informed player.*[14]

*Proof.* See Appendix. □

[13]Recall $G_{A+B}(p)$ is a zero-sum game. It is different from $\mathcal{G}(p)$, which is the 3-player model we analyse.

[14]The generalization of Lemma 3.6 to a model of one informed player and $n$ uninformed players – as can be readily checked in the proof in the Appendix – is straightforward.

**Notation.** If $x := (x_i)_{i\in I} \in \mathbb{R}^I$, then $x^T$ denotes the transpose of row vector $x$.

**Example 3.8.** Consider the game $\mathcal{G}(p^0)$ defined in Example 2.2. We first construct a safe and independent joint-plan with vector payoff $(\alpha, -\beta_A, -\beta_B)$ such that $\alpha \cdot p^0 = \mathrm{Cav}(\bar{h})(p^0) = 1 + 3/16$ and then construct equilibrium strategies associated to the joint-plan. The letter "$U$" will refer to the pure strategy of player 1 in either stage game corresponding to the first row. Analogously, "$D$" will stand for the pure strategy of player 1 corresponding to the second row. The letters "$L$", "$M$" and "$R$" denote the pure strategies of the column player corresponding to the leftmost, middle and right columns, respectively, in matrices $B^k, k = 1, 2$. We use the same letters "$L$" and "$R$" to denote left and right columns in matrices $A^k, k = 1, 2$.

The set of signals is $S = \{(i^1_A, i^1_B), (i'^1_A, i'^1_B)\} = \{(U, U), (D, D)\}$. The signaling strategy is defined by: $x^{1,1}(U, U) = 1/4$ and $x^{1,1}(D, D) = 3/4$; also $x^{2,2}(U, U) = 3/4$ and $x^{2,2}(D, D) = 1/4$. Recall that $\mathrm{supp}(p^0) = \{(1, 1), (2, 2)\}$. In case Nature chooses state $k \in \{1, 2\}$ in both games, the informed player uses the signaling strategy $x^{k,k}$. In case $x^{k,k}$ draws $(U, U)$, then the uninformed players can observe it and update $p^0$ to posterior $p(U, U)$ given by $p^{1,1}(U, U) = 1/4$ and $p^{2,2}(U, U) = 3/4$. In case $(D, D)$ is chosen, the induced posterior is $p(D, D)$ given by $p^{1,1}(D, D) = 3/4$ and $p^{2,2}(D, D) = 1/4$. After the signal realizes, players proceed to play the corresponding "contract", as defined in the paragraph below.

Define contract $\gamma^{(U,U)}_A = \sigma^U_A \bigotimes \tau^U_A$ where $\sigma^U_A = (1/4, 3/4)$ – the first entry of this row vector corresponds to the probability of playing "$U$" and the second to the probability of playing "$D$" – and $\tau^U_A = (3/4, 1/4)^T$, where the first entry corresponds to the probability the uninformed player plays "$L$" and the second entry the probability the uninformed player plays "$R$". Notice that $\tau_A$ is optimal at $\bar{A}(p(U, U)_A)$. Define also $\gamma^{(D,D)}_A = \sigma^D_A \bigotimes \tau^D_A$ where $\sigma^D_A = (3/4, 1/4)$ and $\tau^D_A = (1/4, 3/4)^T$, which is optimal at $\bar{A}(p(D, D)_A)$. The contract $\gamma^{(\mathrm{U,U})}_B$ is defined by strategies $\sigma^{\mathrm{U}}_B = (0, 1)$ and $\tau^{\mathrm{U}}_B = (1/2, 0, 1/2)^T$, which is optimal at $\bar{B}(p(\mathrm{U,U})_B)$. The contract $\gamma^D_B$ is defined by strategies $\sigma^D_B = (1, 0)$ and $\tau^D_B = (0, 1/2, 1/2)^T$, which is optimal $\bar{B}(p(D, D)_B)$.

Using our previously defined notation, we have that $\alpha^1_A := \alpha^1_A(U, U) = \alpha^1_A(D, D) = 3/16$ and $\alpha^1_B := \alpha^1_B(U, U) = \alpha^1_B(D, D) = 1$. Similarly, $\alpha^2_A := \alpha^2_A(U, U) = \alpha^2_A(D, D) = 3/16$ and $\alpha^2_B := \alpha^2_B(U, U) = \alpha^2_B(D, D) = 1$. Then $\alpha^{11} = \alpha^1_A + \alpha^1_B = 1 + 3/16$ and $\alpha^{22} = \alpha^2_A + \alpha^2_B = 1 + 3/16$. Therefore $\alpha = (\alpha^{11}, \alpha^{22})$, which implies that $\alpha \cdot q \geq \bar{h}(q)$, for all $q \in [0, 1]$, which is condition (3) of Lemma 3.6. Condition (2) is also satisfied, since the informed player is indifferent to any contract played after each signal. Notice also that since $\beta_A(U, U) = \sum_{k_A \in K_A} p(U, U)^{k_A}_A \alpha^{k_A}_A$ and $\tau^U_A$ is optimal at $\bar{A}(p(U, U)_A)$, we have that $\beta_A(U, U) \leq v_A(p(U, U)_A) \leq \mathrm{Cav}(v_A)(p(U, U)_A)$; similarly $\beta_B(U, U)$ satisfies analogous conditions, as well as $\beta_A(D, D)$ and $\beta_B(D, D)$. This implies condition (1) of Lemma 3.6 is satisfied. Finally, notice that the ex-ante payoff of the joint-plan is $\alpha \cdot (1/2, 1/2) = 1 + 3/16 = \mathrm{Cav}(\bar{h})(p^0)$.

We now define the equilibrium strategies in $\mathcal{G}(p^0)$ that are induced by the joint-plan. In the first stage player 1 plays the signaling strategy defined by $x$: in case state $(1, 1)$ realizes, player 1 plays according to $x^{1,1}$ and analogously for state $(2, 2)$. This induces posteriors as defined above. If $(U, U)$ realizes as a signal in the first stage of the game, then in the first block of 2 stages following this realization, players 1 and 3 will play $(D, L)$ in the first period and then switch to $(D, R)$ for the next period and repeat this pattern forever in every following block of 2 periods. Let this sequence of deterministic moves be $h^{(U,U)}_{B,\infty} = (i^n_B, j^n_B)_{n\geq 2}$. Thus the payoff of playing this sequence to player 1 is:

$$\frac{1}{N}\sum_{n=2}^{N} B^{k_B}_{i^n_B,j^n_B} \to \sigma^U_B B^{k_B}\tau^U_B = \alpha^{k_B}_B(\text{U,U}) = 1, \forall k_B = 1,2$$

as $N \to +\infty$. If player 3 deviates from this path of play, he is punished by player 1, who plays the optimal strategy $G_B(p(U,U)_B)$ against player 3 forever. Because the expected payoff of player 3 of following this path is $-\beta_B(\text{U,U})$ and because it satisfies condition (1) $\beta_B(U,U) \leq \text{Cav}(v_B)(p(U,U)_B)$ of Lemma 3.6, such a deviation cannot be profitable for player 3.

Analogously, players 1 and 2, in the first block of two stages after $(U,U)$ realizes, play $(U,M)$, and then play $(U,R)$ and repeat this pattern forever. Thus, if $h^{(U,U)}_{A,\infty} = (i^n_A, j^n_A)_{n\geq 2}$ denotes this sequence, we have that

$$\frac{1}{N}\sum_{n=2}^{N} A^{k_A}_{i^n_A,j^n_A} \to \sigma^U_A A^{k_A}\tau^U_A = \alpha^{k_A}_A(U,U) = 1, \forall k_A = 1,2$$

as $N \to +\infty$.

If player 2 deviates, he is punished by player 1, who plays the optimal strategy of $G_A(p(U,U)_A)$. Now, if player 1 deviates from his prescribed moves in either sequence $h^{(U,U)}_{B,\infty}$ or $h^{(D,D)}_{A,\infty}$, then both players 2 and 3 can observe it and play the approachability strategy of game $G_{A+B}(p^0)$. Because $\alpha$ satisfies condition (3) of Lemma 3.6 as we showed, a deviation by player 1 is not profitable for him. Other sequences can be defined similarly for other signals, giving the payoffs $\alpha^{k_B}_B(D,D), \forall k_B \in K_B$ and $\alpha^{k_A}_A(D,D), \forall k_A \in K_A$. The punishments in case of deviations of the deterministic path of play are the same as we defined. Finally, if in the first stage player 1 does not play a signal (e.g., $(U,D)$), such a deviation is observable by players 2 and 3, and they punish him with the approachability strategy of $G_{A+B}(p^0)$.

For the existence result of Theorem 3.1, we construct a uniform equilibrium with specific strategies that allow us to calculate the equilibrium payoff explicitly. Proposition 3.9 below shows that if we stick to strategies of the same class used in the construction (the ones inducing safe and independent joint-plans) it follows that the informed player cannot do better than $\text{Cav}(\bar{h})(p)$ in $\mathcal{G}(p)$.

**Proposition 3.9.** *If $(\sigma, \tau_A, \tau_B)$ is an equilibrium of $\mathcal{G}(p)$ associated to a safe and independent joint-plan, then the ex-ante equilibrium payoff for the informed player is $\text{Cav}(\bar{h})(p)$.*

*Proof.* See Appendix. □

## 4. Equilibrium Payoff Characterization and Folk Theorem

In this section we provide a characterization of the uniform equilibrium payoffs of the informed player in the game $\mathcal{G}(p)$ (Theorem 4.2). We introduce the class of *locally nonrevealing zero-sum games with lack of information on one side*, which is the class of games that allows for a precise characterization. Subsection 4.1 presents the proof of Theorem 4.2; the proof of the lemmas in this subsection are left to the Appendix.

**Definition 4.1.** An infinitely repeated zero-sum game with lack of information on one-side $G_A(p)$ is *locally nonrevealing at $p$* whenever there exist $(\alpha_i)^k_{i=1}$, $\alpha_i > 0, \forall i = 1,...,k$ with $\sum_{i=1}^k \alpha_i = 1$ and $(p_i)^k_{i=1}$, $p_i \in$

$\Delta(K_A), \forall i = 1, ..., k$ such that $\sum_{i=1}^{k} \alpha_i p_i = p$, $\mathrm{Cav}(v_A)(p) = \sum_{i=1}^{k} \alpha_i v_A(p_i)$ and for some $i_0 \in \{1, ..., k\}$ we have $p_{i_0} \in \mathrm{int}\Delta(K_A)$.

The theory of zero-sum games with lack of information on one side[15] tells us that there is always an optimal strategy for the informed player in $G_A(p_A)$ that works as follows: for finitely many stages, the informed player uses his actions to signal about the underlying states, just like we saw in Example 2.2. The uninformed player, depending on the observed actions, updates his prior probability $p$ to one of finitely many posterior probabilities $p_i, i = 1, ..., k$ where $\mathrm{Cav}(v_A)(p) = \sum_{i=1}^{k} \alpha_i v_A(p_i)$. This implies in particular that $v_A(p_i) = \mathrm{Cav}(v_A)(p_i), \forall i = 1, ..., k$. After the updating occurs to $p_i$, the informed player plays the optimal strategy of the zero-sum game $\bar{A}(p_i)$ forever, which guarantees him $v_A(p_i)$. Therefore, with this strategy the informed player can guarantee $\mathrm{Cav}(v_A)(p)$. The property stated in Definition 4.1 is therefore *equivalent* to the existence of an optimal strategy for the informed player where some posterior probability $p_{i_0}$ has full support. We provide a robustness result regarding payoff perturbations for property "locally nonrevealing" in the online appendix.

For $p \in \Delta(K_A \times K_B)$, let

$$I(p) = [\mathrm{Cav}(\bar{h})(p), \mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)].$$

We call $\mathrm{Cav}(\bar{h})(p)$ the *lower end of* $I(p)$ and $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$ the *upper end of* $I(p)$.

**Theorem 4.2.** *The following two statements hold:*

1. *The ex-ante equilibrium payoffs of player 1 in $\mathcal{G}(p)$ are a subinterval of $I(p)$, which contains $\mathrm{Cav}(\bar{h})(p)$.*
2. *If $G_A(p_A)$ and $G_B(p_B)$ are locally nonrevealing at $p_A$ and $p_B$ respectively, then $I(p)$ is the set of ex-ante equilibrium payoffs of player 1 in $\mathcal{G}(p)$.*

Theorem 4.2 tells us that whenever the optimal strategy in each zero-sum game is such that the informed player does not have to exclude any of the possible states (at least in one induced posterior), this implies he can attain the highest payoff in $\mathcal{G}(p)$ in equilibrium (the upper end of $I(p)$), no matter what correlation exists between the zero-sum games. Note that the requirement of "local nonrevelation" of (2) in Theorem 4.2 is a requirement in each zero-sum game separately – which is simple to check once $\mathcal{G}(p)$ is given. This requirement is *orthogonal* to correlation i.e., it is based solely on the payoff structure of each zero-sum game and not on the existence of correlation across states of each zero-sum sum repeated game.

4.1. **Proof of Theorem 4.2.** The proof of Theorem 4.2 follows directly from Lemmas 4.7 and 4.8 below. We start by introducing necessary concepts for the statement and proof of these Lemmas: first, we identify the cases where the interval $I(p)$ is nondegenerate. These are the interesting cases, since whenever $I(p)$ is degenerate – i.e., $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B) = \mathrm{Cav}(\bar{h})(p)$ – it follows that if player 1 plays the optimal strategy of $G_{A+B}(p)$ and players 2 and 3 play the optimal strategy of $G_A(p_A)$ and $G_B(p_B)$ respectively, then the profile of such strategies is an equilibrium with payoff $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B) = \mathrm{Cav}(\bar{h})(p)$. Second, we introduce the notion of nonrevealing loose joint-plan, from which we will construct strategies attaining the upper end of $I(p)$, whenever the conditions of (2) Theorem 4.2 are satisfied.

**Definition 4.3.** Let $p \in \Delta(K_A \times K_B)$. We say $G_A(p_A)$ and $G_B(p_B)$ are *perfectly aligned* when there is an optimal strategy of the informed player in $G_{A+B}(p)$ that induces optimal strategies in $G_A(p_A)$ and $G_B(p_B)$.

[15]See Sorin (2002).

**Proposition 4.4.** *Let $p \in \Delta(K_A \times K_B)$. The interval $I(p)$ is nondegenerate if and only if $G_A(p_A)$ and $G_B(p_B)$ are not perfectly aligned.*[16]

*Proof.* See Appendix. □

For an intuition of the proof of Proposition 4.4, when a game has perfectly aligned strategies the informed player can guarantee $\mathrm{Cav}(v_A)(p_A)$ in $G_A(p_A)$ and $\mathrm{Cav}(v_B)(p_B)$ in $G_B(p_B)$ by playing the optimal strategy of $G_{A+B}(p)$. Therefore $\mathrm{Cav}(\bar{h})(p) = \mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$. If $I(p)$ on the other hand is degenerate then this implies the informed play can guarantee $\mathrm{Cav}(\bar{h})(p) = \mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$ by playing his optimal strategy in $G_{A+B}(p)$. Therefore, this optimal strategy induces optimal strategies in each zero-sum game $G_A(p_A)$ and $G_B(p_B)$. The game $\mathcal{G}(p^0)$ of Example 2.2 does not have perfectly aligned strategies and therefore the associated interval $I(p^0)$ is nondegenerate: we have $\mathrm{Cav}(v_A)(p^0_A) + \mathrm{Cav}(v_B)(p^0_B) = 1/4 + 1 > 1 + 3/16 = \mathrm{Cav}(\bar{h})(p^0)$.

Given the result in Proposition 3.9, in order to construct equilibrium strategies that induce a payoff equal to $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$ for the informed player, we have to use strategies of a different class than the ones induced by safe and independent joint-plans. We therefore introduce the class of *nonrevealing loose joint-plans*. Formally, nonrevealing loose joint-plans are defined in each zero-sum game $G_A(p_A)$ and $G_B(p_B)$ individually. If each of the zero-sum games possesses the property of Definition 4.1, then it is possible to construct equilibrium strategies in $\mathcal{G}(p)$ attaining $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$ as payoff for the informed player.

**Definition 4.5.** Given $p \in \Delta(K_A)$, we define a *nonrevealing loose joint-plan* in $G_A(p)$ by a tuple $(\mathcal{O}_{\mathcal{A}}, (\lambda^A_s)_{s\in\mathcal{O}_A}, (\gamma^A_s)_{s\in\mathcal{O}_A})$, where

- (lottery outcomes) $\mathcal{O}_A$ is a finite set.
- (lottery probabilities) $\lambda^A_s \geq 0, \forall s \in \mathcal{O}_A$ and $\sum_{s\in\mathcal{O}_A} \lambda^A_s = 1$.
- (contracts) $(\gamma^A_s)_{s\in\mathcal{O}_A}$ and $\gamma^A_s = \sigma^A_s \bigotimes \tau^A_s$.

The intuition for a nonrevealing loose joint-plan is: in the zero-sum repeated game $G_A(p)$, players 1 and 2 first play a *jointly controlled lottery* in multiple outcomes[17], which will run for finitely many stages. The jointly controlled lottery works like a public randomization device that can be endogenously generated by the players with their own actions in the game. If the outcome of the lottery is $s \in \mathcal{O}_A$, then the players – just like in an independent joint-plan – play the contract $\gamma^A_s$, which is a deterministic sequence of moves whose asymptotic frequency is $\gamma^A_s$. Because the jointly controlled lottery procedure reveals no information, the path of play induced by a nonrevealing loose joint plan will also reveal no information.

**Notation.** For $\ell = A, B$, let $(\mathcal{O}_\ell, (\lambda^\ell_s)_{s\in\mathcal{O}_\ell}, (\gamma^\ell_s)_{s\in\mathcal{O}_\ell})$ be a nonrevealing loose joint-plan in $G_\ell(p_\ell)$, and $\alpha^{k_\ell} := \sum_{s\in\mathcal{O}_\ell} \lambda^\ell_s (\sum_{i_\ell, j_\ell} \ell^{k_\ell}_{i_\ell, j_\ell} \gamma^\ell_s(i_\ell, j_\ell))$. Define $\alpha^\ell := (\alpha^{k_\ell})_{k_\ell \in \mathrm{supp}(p_\ell)}$ and $\beta_\ell := \sum_{k_\ell \in K_\ell} p^{k_\ell} \alpha^{k_\ell}$. The vector $(\alpha^\ell, -\beta_\ell)$ is called the vector of payoffs of the nonrevealing loose joint-plan in $G_\ell(p_\ell)$.

[16] Games $\mathcal{G}(p)$ where $I(p)$ is nondegenerate are robust to payoff perturbations. We prove this in the online appendix.

[17] Jointly controlled lotteries were defined by Aumann, Maschler and Stearns (1995) for two outcomes but can be easily generalized to multiple (finite) outcomes: for instance, if the outcome set is $O_A = \{o_1, o_2, o_3\}$, with probabilities $\lambda_1, \lambda_2$ and $\lambda_3$ for $o_1, o_2$ and $o_3$ respectively, then players can run first a jointly controlled lottery in two outcomes, say $o_1$ and $\beta_2$ with probabilities $\lambda_1$ and $(1 - \lambda_1)$. Aumann, Maschler and Stearns (1995)(p.274) showed that it takes finitely many stages to implement this lottery. If $\beta_2$ realizes, then run another jointly controlled lottery with outcomes $o_2$ and $o_3$, with probabilities $\frac{\lambda_2}{1-\lambda_1}$ and $\frac{\lambda_3}{1-\lambda_1}$, respectively.

**Lemma 4.6.** *For $p \in \Delta(K_A \times K_B)$, let $(\mathcal{O}_A, (\lambda_s^A)_{s\in\mathcal{O}_A}, (\gamma_s^A)_{s\in\mathcal{O}_A})$ be a nonrevealing loose joint-plan in $G_A(p_A)$ and $(\mathcal{O}_\mathcal{B}, (\lambda_s^B)_{s\in\mathcal{O}_B}, (\gamma_s^B)_{s\in\mathcal{O}_B})$ a nonrevealing loose joint-plan in $G_B(p_B)$. The following statements hold:*

(1) *If (i) $\alpha^A \cdot q_A \geq v_A(q_A), \forall q_A \in \Delta(supp(p_A))$ and (ii) $\beta_A \leq Cav(v_A)(p_A)$, then there exists an equilibrium $(\sigma_A, \tau_A)$ in $G_A(p_A)$ with $\alpha(\sigma_A, \tau_A) = \alpha^A$ and $\beta^A(\sigma_A, \tau_A) = -\beta_A$.*
(2) *If (i) $\alpha^B \cdot q_B \geq v_B(q_B), \forall q_B \in \Delta(supp(p_B))$ and (ii) $\beta_B \leq Cav(v_B)(p_B)$, then there exists an equilibrium $(\sigma_B, \tau_B)$ in $G_B(p_B)$ with $\alpha(\sigma_B, \tau_B) = \alpha^B$ and $\beta^B(\sigma_B, \tau_B) = -\beta_B$.*
(3) *If (1.i) $\alpha^A \cdot q_A \geq v_A(q_A), \forall q_A \in \Delta(supp(p_A))$, (1.ii) $\beta_A \leq Cav(v_A)(p_A)$, (2.i) $\alpha^B \cdot q_B \geq v_B(q_B), \forall q_B \in \Delta(supp(p_B))$ and (2.ii) $\beta_B \leq Cav(v_B)(p_B)$, then there exists an equilibrium $(\sigma, \tau_A, \tau_B)$ in $\mathcal{G}(p)$ such that $\alpha(\sigma, \tau_A, \tau_B) = (\alpha^{k_A} + \alpha^{k_B})_{(k_A,k_B)\in supp(p)}$, $\beta^A(\sigma, \tau_A, \tau_B) = -\beta_A$ and $\beta^B(\sigma, \tau_A, \tau_B) = -\beta_B$.*

*Proof.* See Appendix. □

Lemma 4.6 is the analogous version of Lemma 3.6 for nonrevealing loose joint-plans. Conditions (i), (ii), (1.i), (1.ii), (2.i), (2.ii) in this theorem have the same role as conditions (1) and (3) in Lemma 3.6 – they guarantee the existence of strategies to punish a player that deviates from the path of play given by a contract.

**Lemma 4.7.** *Let $p \in \Delta(K_A \times K_B)$ and assume $G_A(p_A)$ and $G_B(p_B)$ are both locally nonrevealing games at $p_A$ and $p_B$ respectively. Then the upper end of $I(p)$ is an equilibrium payoff of the informed player in $\mathcal{G}(p)$.*

*Proof.* See Appendix. □

For the proof of Lemma 4.7 we use nonrevealing loose joint-plans to construct equilibria attaining the upper end of $I(p)$. The crucial property of nonrevealing loose joint-plans is that the path of play induced by such plans reveals no information. This allows the informed player to overcome the adverse effects of information spillover. Lemma 4.7 is the central result in the proof of Theorem 4.2. Lemma 4.8 is just a restatement of (1) Theorem 4.2 and Corollary 4.9 finishes the proof of (2) Theorem 4.2.

**Lemma 4.8.** *The ex-ante uniform equilibrium payoff set of the informed player in $\mathcal{G}(p)$ is a subinterval of $I(p)$ that contains the lower end $Cav(\bar{h})(p)$.*

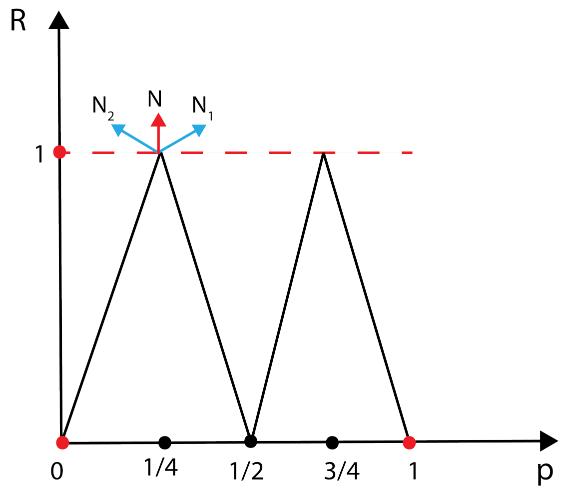
*Proof.* See Appendix. □

**Corollary 4.9.** *[Folk Theorem] Let $p \in \Delta(K_A \times K_B)$ with $G_A(p_A)$ and $G_B(p_B)$ not perfectly aligned and locally nonrevealing at $p_A$ and $p_B$ respectively. Then $I(p)$ is nondegenerate and is the set of ex-ante equilibrium payoffs of the informed player.*

*Proof.* The no perfect alignment assumption gives that $I(p)$ is nondegenerate, by Proposition 4.4. The local nonrevelation condition at $p_A$ and $p_B$ implies that the upper end of $I(p)$ is an ex-ante equilibrium payoff of the informed player, by Lemma 4.7. Applying now Lemma 4.8 finishes the proof. □

4.2. **Example.** The example below ilustrates the ideas contained in the proof of Theorem 4.2. Let the game $\mathcal{G}(p)$ be given by stage payoffs defined in Example 2.2. We construct nonrevealing loose joint-plans for $\mathcal{G}(p^0)$ attaining $\mathrm{Cav}(v_A)(p_A^0) + \mathrm{Cav}(v_B)(p_B^0)$. The construction process parallels the construction in the proof of Theorem 4.2. First notice that the zero-sum games $G_A(p_A^0)$ and $G_B(p_B^0)$ in this example are both locally

FIGURE 3. Graph of $v_B$




nonrevealing at $p_A^0$ and $p_B^0$ respectively, since the nonrevealing value function of game $G_A(p_A^0)$ is strictly concave and in $G_B(p_B^0)$ there is an optimal strategy for the informed player – constructed in Example 2.2 – where the induced posterior probabilities have full support. We start with the construction of a nonrevealing loose joint-plan $(\mathcal{O}_B, (\lambda_s^B)_{s\in\mathcal{O}_B}, (\gamma_s^B)_{s\in\mathcal{O}_B})$ in game $G_B(p_B^0)$ that will satisfy (2.i) and (2.ii) in (3) of Lemma 4.6. The construction derives from geometric properties of the nonrevealing value function $v_B$.

Since $p_B^0 = 1/2$ (the probability of state 1), we saw in Example 2.2 that the informed player has an optimal strategy where he induces posterior probabilities equal to 1/4 or 3/4.

Let $\tau_r = (0,0,1)$, $\tau_l = (1,0,0)$ and $s_b = (0,1)$. For $q \in [1/4, 1/2)$, the nonrevealing value function $v_B$ is given by $(s_b B^1 \tau_r^T, s_b B^2 \tau_r^T) \cdot (q, 1-q) = (-2, 2) \cdot (q, 1-q)$; for $q \in [0, 1/4]$, it is given by $(s_b B^1 \tau_l^T, s_b B^2 \tau_l^T) \cdot (q, 1-q) = (4, 0) \cdot (q, 1-q)$. Looking now at the graph of $v_B$ as depicted in Figure 3, we have that $\forall (q, y) \in [1/4, 1/2) \times \mathbb{R}, [(q, y) \in \text{Graph}(v_B) \iff (-4, -1) \cdot (q, y) = -2]$. Similarly, $\forall (q, y) \in [0, 1/4] \times \mathbb{R}, [(q, y) \in \text{Graph}(v_B) \iff (4, -1) \cdot (q, y) = 0]$. The normal vectors $N_1 = -(-4, -1)$ and $N_2 = -(4, -1)$ are depicted in Figure 3 at the point $(1/4, 1) \in \mathbb{R}^2$. Notice now that the vector $N = (1/2)N_1 + (1/2)N_2$ defines a hyperplane $H = \{(q, y) \in (0, 1) \times \mathbb{R} | -N \cdot (q, y) = 1\}$ that is "above" the graph of the nonrevealing value function $v_B$. The hyperplane $H$ is the graph of the affine function $[(1/2)(s_b B^1 \tau_l^T, s_b B^2 \tau_l^T) + (1/2)(s_b B^1 \tau_r^T, s_b B^2 \tau_r^T)] \cdot (q, 1-q) = (1, 1) \cdot (q, 1-q) = 1$. Recall that $\text{Cav}(v_B)(q) = 1, \forall q \in [1/4, 3/4]$, so we have in particular that $[(1/2)(s_b B^1 \tau_l^T, s_b B^2 \tau_l^T) + (1/2)(s_b B^1 \tau_r^T, s_b B^2 \tau_r^T)] \cdot (q, 1-q) \geq v_B(q), \forall q \in [0, 1]$ and $[(1/2)(s_b B^1 \tau_l^T, s_b B^2 \tau_l^T) + (1/2)(s_b B^1 \tau_r^T, s_b B^2 \tau_r^T)] \cdot (1/2, 1/2) = 1 = \text{Cav}(v_B)(p_B^0)$.

From these formal observations, we can now define the nonrevealing loose joint-plan $(\mathcal{O}_B, (\lambda_s^B)_{s\in\mathcal{O}_B}, (\gamma_s^B)_{s\in\mathcal{O}_B})$ as follows: let $o_1 := \{(U, L), (D, M)\}$, where $(U, L)$ are the moves in the first stage of players 1 and 3 corresponding to the first row for player 1 and leftmost column for player 3; analogously, $(D, M)$ are the moves in the first stage corresponding to second row for player 1 and middle column for player 3. Let $o_2 = \{(D, L), (U, M)\}$. The set of outcomes $\mathcal{O}_B$ is defined as the set containing $o_1$ and $o_2$. We now define a jointly controlled lottery in the first stage that will draw $o_1$ with probability $\lambda_{o_1}^B = 1/2$ and $o_2$ with probability $\lambda_{o_2}^B = 1/2$ as well as the strategies to be played after the jointly controlled lottery is implemented. First, player 1 will randomize in the first stage between "$U$" and "$D$" with equal probabilities; player 3 will randomize in the first stage with equal probabilities between "$L$" and "$M$". In the first stage, in case either of the profiles in $o_1$ is drawn, then players 1 and 3 will play $(s_b, \tau_l)$ at each stage, forever; in case either of the profiles in $o_2$ realizes, then players 1 and 3 will play $(s_b, \tau_r)$ at each stage forever. If player 3 deviates to playing "$R$" in the first stage, player 1 plays the optimal strategy of the zero-sum game $G_B(p_B^0)$.

Because of the way the lottery in the first stage is constructed, no player can unilaterally change the probabilities with which $o_1$ or $o_2$ is drawn. If players 1 and 3 follow the lottery and the specified path of play thereafter, player 1 obtains a vector of payoffs equal to $[(1/2)(s_b B^1 \tau_l^T, s_b B^2 \tau_l^T) + (1/2)(s_b B^1 \tau_r^T, s_b B^2 \tau_r^T)] = (1,1)$ and ex-ante payoff $[(1/2)(s_b B^1 \tau_l^T, s_b B^2 \tau_l^T) + (1/2)(s_b B^1 \tau_r^T, s_b B^2 \tau_r^T)] \cdot (1/2, 1/2) = 1 = \mathrm{Cav}(v_B)(p_B^0)$.

Hence, we define formally $\mathcal{O}_B = \{o_1, o_2\}$; the contracts are $\gamma_{o_1}^B := s_b \bigotimes \tau_l$ and $\gamma_{o_2}^B := s_b \bigotimes \tau_r$. It follows that $\alpha^B \cdot (q, 1-q) = [(1/2)(s_b B^1 \tau_l^T, s_b B^2 \tau_l^T) + (1/2)(s_b B^1 \tau_r^T, s_b B^2 \tau_r^T)] \cdot (q, 1-q) = (1,1) \cdot (q, 1-q) = 1 \geq v_B(q), \forall q \in [0,1]$, which is condition (2.i) of Lemma 4.6; condition (2.ii) is also satisfied since $\beta_B = [(1/2)(s_b B^1 \tau_l^T, s_b B^2 \tau_l^T) + (1/2)(s_b B^1 \tau_r^T, s_b B^2 \tau_r^T)] \cdot (1/2, 1/2) = 1 = \mathrm{Cav}(v_B)(p_B^0)$.

We now define $(\mathcal{O}_A, (\lambda_s^A)_{s \in \mathcal{O}_A}, (\gamma_s^A)_{s \in \mathcal{O}_A})$. In this case, we consider only one outcome for the lottery $\mathcal{O}_A = \{o_1\}$ because no real randomization is necessary: players can begin to play a single contract after the first stage that will satisfy the assumption of (1.i) and (1.ii) of (3) in Lemma 4.6. For completeness, we define $\mathcal{O}_A = \{o_1\}$ and $o_1$ corresponds to any move by player 1 in the first stage of the game.

We now define the contract to be played after the first stage in $G_A(p_A^0)$. Consider the stage game strategies $\sigma_A = (1/2, 1/2)$ and $\tau_A = (1/2, 1/2)$ and define $\gamma_{o_1}^A := \sigma_A \bigotimes \tau_A$. Notice that $\alpha^A = (\sigma_A A^1 \tau_A^T, \sigma_A A^2 \tau_A^T) = (1/4, 1/4)$. This implies that $(\sigma_A A^1 \tau_A^T, \sigma_A A^2 \tau_A^T) \cdot (q, 1-q) = 1/4 \geq q(1-q) = v_A(q)$. This implies condition (1.i) of (3) in Lemma 4.6 is satisfied. Notice also that $\beta_A = \alpha^A \cdot (1/2, 1/2) = 1/4 = \mathrm{Cav}(v_A)(1/2)$, which implies (1.ii) of (3) in Lemma 4.6.

Similarly as we did in Example 3.8, the equilibrium strategies in $G_A(p_A^0)$ can be defined as follows: for every block of 4 periods, players 1 and 2 will play $(U, L)$ in the first period, $(U, R)$ in the second period, $(D, R)$ in the third period and $(D, L)$ in the last period. The asymptotic empirical distribution of this path of play is the distribution $\sigma_A \bigotimes \tau_A$ and the payoff is $\alpha^A$. We already defined the path of play in game $G_B(p_B^0)$, so what remains is to define the strategies in case players deviate from the path of play. In case player 1 deviates from the path of play in either of the games, the deviation is observable and the uninformed players play the approachability strategy of $G_{A+B}(p^0)$. The vector of payoffs of player 1 when everyone follows the path of play is $\alpha = (\sigma_A A^{k_A} \tau_A^T + \sigma_B B^{k_B} \tau_B^T)_{(k_A, k_B) \in \mathrm{supp}(p)}$. For each $q \in [0,1]$, it implies that $\alpha \cdot q = \alpha_A \cdot q_A + \alpha_B \cdot q_B \geq v_A(q_A) + v_B(q_B) = \bar{h}(q)$. This shows that $\alpha$ can indeed be approached by players 2 and 3, which implies that there is no profitable deviation for player 1. Now, in case player 2 deviates from the prescribed path of play, player 1 plays the optimal strategy of $G_A(p_A^0)$: this optimal strategy holds player 2 at most at $-\mathrm{Cav}(v_A)(p_A^0) = -1/4$, which shows that a deviation is not profitable for player 2. The analogous reasoning applies in case of a deviation of player 3. The ex-ante payoff to player 1 of the nonrevealing loose joint plan is then $1 + 1/4 = \mathrm{Cav}(v_A)(p_A^0) + \mathrm{Cav}(v_B)(p_B^0)$.

The key idea in the construction above is to identify strategies of the stage games such that they induce contracts whose convex combinations define payoffs $\alpha^A$ and $\alpha^B$ which are the "supergradients" $(1/4, 1/4)$ of $\mathrm{Cav}(v_A)$ at $p_A^0$ and $(1,1)$ of $\mathrm{Cav}(v_B)$ at $p_B^0$, respectively. The convex combination can then be implemented through a jointly controlled lottery, which by design does not reveal any information.

The following example shows that the condition in (2) of Theorem 4.2 is not necessary for an equilibrium to pay the upper end of $I(p)$ for the informed player. In this example we have $I(p)$ nondegenerate – $\mathrm{Cav}(\bar{h})(p^0) = 0 < 1/4 = \mathrm{Cav}(v_A)(p_A^0) + \mathrm{Cav}(v_B)(p_B^0)$ – and the upper end of $I(p)$ is attainable as an ex-ante equilibrium-payoff for the informed player, but $G_B(p_B^0)$ is not locally nonrevealing at $p_B^0$.

**Example 4.10.** Let

$$p^0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \; A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Consider $\sigma_B = (1/2, 1/2)$ and $\tau_B = (1/2, 1/2)$ as stage game strategies defining the contract $\gamma_B$ in $G_B(p_B^0)$. The vector of payoffs $\alpha^B$ of contract $\gamma_B$ is $(1/4, 1/4)$. So, $\alpha^B \cdot q = \sigma_B \bar{B}(q) \tau_B^T = 1/4 \geq v_B(q)$, $\forall q \in [0,1]$. Consider now strategies $\sigma_A = (1, 0)$ and $\tau_A = (0, 1)$ and let the contract in $G_A(p_A^0)$ be $\gamma_A := \sigma_A \bigotimes \tau_A^T$. Then the vector of payoffs $\alpha_A$ of this contract is $(0, 0)$, so $\alpha^A \cdot q = \sigma_A \bar{A}(q) \tau_A^T = \mathrm{Cav}(v_A)(q) = 0, \forall q \in [0,1]$. These two contracts define nonrevealing loose joint-plans satisfying conditions (3) of Lemma 4.6 – which implies there is an equilibrium whose ex-ante payoff is $1/4 = \mathrm{Cav}(v_A)(p_A^0) + \mathrm{Cav}(v_B)(p_B^0)$.

## 5. An Example

In this section, we answer affirmatively (by means of an example) the question of whether there are instances where the interval $I(p)$ is nondegenerate and the upper end of $I(p)$ is not attainable as an ex-ante equilibrium payoff for the informed player. In order to do it, we prove Proposition 5.5, which provides a necessary condition for the upper end of $I(p)$ to be attained as an ex-ante equilibrium payoff for the informed player. Indeed our example allows us to show more: there will be only one ex-ante equilibrium payoff of the informed player, which is the lower end of $I(p)$.

### 5.1. Necessary condition for attainability of upper end of $I(p)$.

We start by defining three martingale processes which were already used by Hart (1985) in his characterization of equilibrium payoffs for two player games.

**Definition 5.1.** Let $\ell_\infty$ be the Banach space of real bounded sequences $x = (x_n)_{n \in \mathbb{N}^*}$. A Banach-Limit[18] is a real functional $\mathcal{L} : \ell_\infty \to \mathbb{R}$ with the following properties:

(1) $\mathcal{L}((\lambda.x_n + \mu.y_n)_{n \in \mathbb{N}^*}) = \lambda.\mathcal{L}((x_n)_{n \in \mathbb{N}^*}) + \mu.\mathcal{L}((y_n)_{n \in \mathbb{N}^*}), \forall \lambda, \mu \in \mathbb{R}$.
(2) $\mathcal{L}((x_{n+1})_{n \in \mathbb{N}^*}) = \mathcal{L}((x_n)_{n \in \mathbb{N}^*})$.
(3) $\liminf_{n \to +\infty} x_n \leq \mathcal{L}((x_n)_{n \in \mathbb{N}^*}) \leq \limsup_{n \to +\infty} y_n$.

[18]The concept of Banach-Limit is introduced in Hart (1985), section 4. See also Dunford and Schwartz (1957), p. 73.

**The Basic Probability Space.** For $t \in \mathbb{N}^*$, denote by $\mathcal{H}_t$ the finite field generated by the elements in $H_t$, with $\mathcal{H}_1$ being the trivial field. Define by $\mathcal{H}_\infty$ the $\sigma$-field generated by $(\mathcal{H}_t)_{t\in\mathbb{N}^*}$. Note that $(\mathcal{H}_t)_{t\in\mathbb{N}^*}$ is an increasing sequence of finite subfields of $\mathcal{H}_\infty$. The space $\Omega = H_\infty \times K_A \times K_B$ is endowed with the $\sigma$-field $\mathcal{H}_\infty \bigotimes 2^{K_A\times K_B}$. Since the probability space is $\Omega$, we will also denote the field generated by $H_t$ on $\Omega$ by $\mathcal{H}_t$. Each profile of strategies $(\sigma, \tau_A, \tau_B)$ and each probability vector $p \in \triangle(K_A \times K_B)$ determine a probability distribution on the space $\Omega$. Recall that we denote by $\mathbb{P}_{\sigma,\tau_A,\tau_B,p}$ this probability measure, $\mathbb{E}_{\sigma,\tau_A,\tau_B,p}$ the expectation with respect to this measure, and $\mathbb{E}^{k_A,k_B}_{\sigma,\tau_A,\tau_B,p}$ the conditional expectation with respect to $\kappa_A \times \kappa_B = (k_A, k_B)$, where $\kappa_A \times \kappa_B$ is a random variable representing Nature's randomization.

Let $\mathcal{L}$ be a Banach-limit. A triple of strategies $(\sigma, \tau_A, \tau_B)$ is an $\mathcal{L}$-equilibrium if:

1. $\mathcal{L}(\mathbb{E}^{k_A,k_B}_{\sigma,\tau_A,\tau_B,p}[\frac{1}{T}\sum_{t=1}^{T}(A^{k_A}_{i^t_A,j^t_B} + B^{k_B}_{i^t_A,j^t_B})]) \geq \mathcal{L}(\mathbb{E}^{k_A,k_B}_{\overline{\sigma},\tau_A,\tau_B,p}[\frac{1}{T}\sum_{t=1}^{T}(A^{k_A}_{i^t_A,j^t_B} + B^{k_B}_{i^t_A,j^t_B})])$, for all $(k_A, k_B) \in \text{supp}(p)$, for all strategies $\overline{\sigma}$ of the informed player 1.
2. $\mathcal{L}(\mathbb{E}_{\sigma,\tau_A,\tau_B,p}[\frac{1}{T}\sum_{t=1}^{T}(-A^{\kappa_A}_{i^t_A,j^t_A})]) \geq \mathcal{L}(\mathbb{E}_{\sigma,\overline{\tau}_A,\tau_B,p}[\frac{1}{T}\sum_{t=1}^{T}(-A^{\kappa_A}_{i^t_A,j^t_A})])$, for all strategies $\overline{\tau}_A$ of the uninformed player 2.
3. $\mathcal{L}(\mathbb{E}_{\sigma,\tau_A,\tau_B,p}[\frac{1}{T}\sum_{t=1}^{T}(-B^{\kappa_B}_{i^t_B,j^t_B})]) \geq \mathcal{L}(\mathbb{E}_{\sigma,\tau_A,\overline{\tau}_B,p}[\frac{1}{T}\sum_{t=1}^{T}(-B^{\kappa_B}_{i^t_B,j^t_B})])$, for all strategies $\overline{\tau}_B$ of the uninformed player 3.

The set of equilibrium profiles is the same under any Banach-Limit $\mathcal{L}$, so there is no loss of generality in fixing a Banach-Limit $L$ throughout (see Section 4.2 in Hart (1985)). A uniform equilibrium of the game $\mathcal{G}(p)$ will automatically satisfy 1, 2, 3, above, so it will be an $L$-equilibrium.

5.2. **Martingale of Posteriors.** Let $(\sigma, \tau_A, \tau_B)$ be a profile of strategies. We define the process of posteriors obtained through Bayesian updating. For $t \in \mathbb{N}^*$, let $p^{k_A,k_B}_t := \mathbb{P}_{\sigma,\tau_A,\tau_B,p}(\kappa_A \times \kappa_B = (k_A, k_B)|\mathcal{H}_t)$ and $p_t := (p^{k_A,k_B}_t)_{k_A,k_B}$.

**Lemma 5.2.** *The sequence $(p_s)_{s\in\mathbb{N}^*}$ is a $\Delta(K_A\times K_B)$-valued martingale with respect to $(\mathcal{H}_s)_{s\in\mathbb{N}^*}$, satisfying:*

1. $p_1 = p$,
2. *There exists $p^\infty$ such that $p_t \to p^\infty$ a.s. as $t \to +\infty$.*

The a.s. limit of the process $(p_t)_{t\in\mathbb{N}^*}$ will be denoted $p^\infty$ and called asymptotic posterior.

*Proof.* The martingale property follows immediately from the definition of $(p_t)_{t\in\mathbb{N}^*}$. Property (1) follows from the fact that $\mathcal{H}_1$ is the trivial $\sigma$-field. Property (2) follows from the martingale convergence theorem. □

5.3. **The Expected Payoff Martingales.** Let $(\sigma, \tau_A, \tau_B)$ be a profile of strategies. For $s \in \mathbb{N}^*$, let $\beta_{A,s} := L(\mathbb{E}[\alpha_T|\mathcal{H}_s])$, where $\alpha_T = \frac{1}{T}\sum\limits_{t=1}^{T} A^{\kappa_A}_{i^t_A,j^t_A}$ and $\beta_{B,s} := L(\mathbb{E}[\beta_T|\mathcal{H}_s])$, where $\beta_T = \frac{1}{T}\sum\limits_{t=1}^{T} B^{\kappa_B}_{i^t_B,j^t_B}$.

These martingales correspond to the process of expected payoffs accumulated by the informed player in each zero-sum repeated game. The martingale property follows from the following fact: For $s \in \mathbb{N}^*$, $X_s = (\mathbb{E}[\alpha_T|\mathcal{H}_s])_{T\in\mathbb{N}^*}$ is a random element taking finitely many values in $\ell_\infty$. By Lemma 4.6 in Hart (1985), it follows that the Banach-limit $L$ commutes with the conditional expectation operator, which implies the martingale property. Given an equilibrium profile $(\sigma, \tau_A, \tau_B)$ of $\mathcal{G}(p)$, a probability space is defined over the product of histories and states. At each stage $s \in \mathbb{N}^*$, according to the information revealed by $\sigma$, the uninformed players have an "assessment" of the payoff they accumulate if they stick to the equilibrium strategies. This is what $\beta_{A,s}$ and $\beta_{B,s}$ represent: they are the negative of the expected payoff for the uninformed players 2 and 3 respectively of playing according to the equilibrium strategies given the information available at stage $s \in \mathbb{N}^*$. The "information available at stage $s$" is captured by the finite field $\mathcal{H}_s$.

**Lemma 5.3.** *Let $(\sigma, \tau_A, \tau_B)$ be a uniform equilibrium with payoff vector $a \in \mathbb{R}^{supp(p)}$ for the informed player in $\mathcal{G}(p)$. The stochastic processes $(\beta_{A,s})_{s\in\mathbb{N}^*}$ and $(\beta_{B,s})_{s\in\mathbb{N}^*}$ are $\mathbb{R}_M$-valued martingales with respect to $(\mathcal{H}_s)_{s\in\mathbb{N}^*}$ satisfying $\beta_{A,1} + \beta_{B,1} = a \cdot p$*

*Proof.* Follows directly from the definition of $(\beta_{A,s})_{s\in\mathbb{N}^*}$ and $(\beta_{B,s})_{s\in\mathbb{N}^*}$ and the fact that both are martingales. □

**Lemma 5.4.** *If $(\sigma, \tau_A, \tau_B)$ is an equilibrium profile then it must satisfy:*

(1) $\beta_{A,s} \leq Cav(v_A)(p_{sA})$*, for all* $s \in \mathbb{N}^*$.
(2) $\beta_{B,s} \leq Cav(v_B)(p_{sB})$*, for all* $s \in \mathbb{N}^*$.

*Proof.* Follows from Propositions 4.40 in Hart (1985). □

**Proposition 5.5.** *Let $(\sigma, \tau_A, \tau_B)$ be an equilibrium of $\mathcal{G}(p^0)$ where the ex-ante payoff of the informed player is $Cav(v_A)(p^0_A) + Cav(v_B)(p^0_B)$. Then $(Cav(v_\ell)(p_{s\ell}))_{s\in\mathbb{N}^*}$ is a martingale with respect to $(\mathcal{H}_s)_{s\in\mathbb{N}^*}$, for each $\ell \in \{A, B\}$.*[19]

*Proof.* First, notice that for each $\ell \in \{A, B\}$ and $k, s \in \mathbb{N}^*$ with $k \leq s$ we have that $\mathrm{Cav}(v_\ell)(p_{k\ell}) = \mathrm{Cav}(v_\ell)(\mathbb{E}[p_{s\ell}|\mathcal{H}_k]) \geq \mathbb{E}[\mathrm{Cav}(v_\ell)(p_{s\ell})|\mathcal{H}_k]$ a.s. – where the equality follows from the fact that the marginal operator is affine and $(p_s)_{s\in\mathbb{N}^*}$ is a martingale, and the inequality follows from Jensen's inequality. Asumme by contradiction that there exist $k, s \in \mathbb{N}^*$ with $k < s$, $\ell_0 \in \{A, B\}$ and an atom $h_k \in \mathcal{H}_k$ such that $\mathrm{Cav}(v_{\ell_0})(p_{k\ell_0}(h_k)) > \mathbb{E}[\mathrm{Cav}(v_{\ell_0})(p_{s\ell_0})|\mathcal{H}_k](h_k)$. It follows that $\mathrm{Cav}(v_{\ell_0})(p^0_{\ell_0}) \geq \mathbb{E}[\mathrm{Cav}(v_{\ell_0})(p_{k\ell_0})] > \mathbb{E}[\mathrm{Cav}(v_{\ell_0})(p_{s\ell_0})] \geq \mathbb{E}[\beta_{\ell_0,s}] = \beta_{\ell_0,1}$, where the first inequality is given by Jensen's inequality, the second by assumption, the third by Lemma 5.4 and the last equality by definition of the martingales of expected payoffs. This then implies that $\mathrm{Cav}(v_A)(p^0_A) + \mathrm{Cav}(v_B)(p^0_B) > \beta_{A,1} + \beta_{B,1}$. Contradiction, since by Lemma 5.3 we have that $\beta_{A,1} + \beta_{B,1} = \mathrm{Cav}(v_A)(p^0_A) + \mathrm{Cav}(v_B)(p^0_B)$. □

The requirement that $(\mathrm{Cav}(v_\ell)(p_{s\ell}))_{s\in\mathbb{N}^*}$ is a martingale for each $\ell \in \{A, B\}$ has a straightforward geometric interpretation: it restricts the regions of $\Delta(K_\ell)$ on which the martingale of posteriors may take values. In Example 5.6 below, the process $(\mathrm{Cav}(v_B)(p_{sB}))_{s\in\mathbb{N}^*}$ is a martingale if only if for each $s \in \mathbb{N}^*$, $p_{sB}$ belongs to the interval $[1/4, 3/4]$ almost surely.

**Example 5.6.** The graph of Figure 4 below is taken from Example 2.2. The prior on state 1 is $p^0 = 1/2$ .

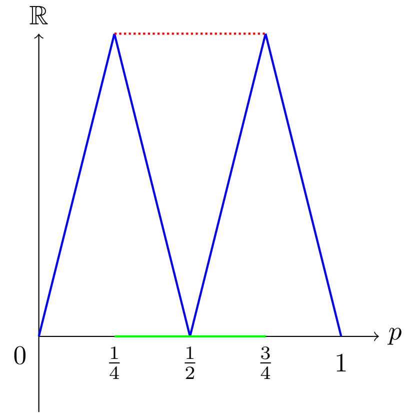
Using the necessary condition of Proposition 5.5, we show an example where the interval $I(p)$ is nondegenerate and the upper end is not attainable - indeed only the lower end is an equilibrium payoff.

**Example 5.7.** Let

$$p^0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

[19] I thank Tristan Tomala for this formulation.

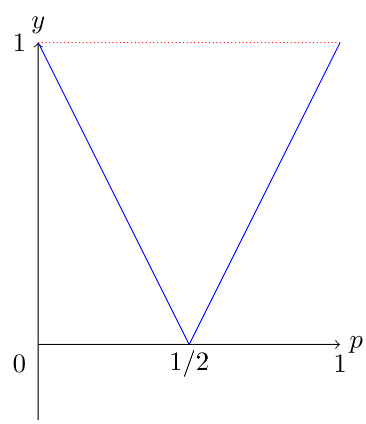
FIGURE 4. Graph of $v_B$ in Example 2.2




$$A^1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

FIGURE 5. Graphs of $\mathrm{Cav}(v_A)$(dotted) and $v_A$(continuous)

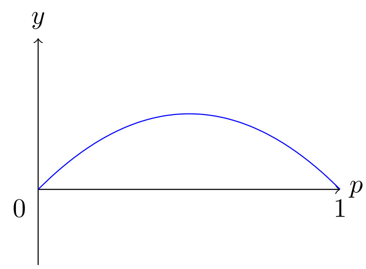



Assume by way of contradiction that $(\sigma, \tau_A, \tau_B)$ is an equilibrium that pays ex-ante $\mathrm{Cav}(v_A)(p^0_A) + \mathrm{Cav}(v_B)(p^0_B)$ for the informed player in $\mathcal{G}(p^0)$. Let $V_A : \triangle(K_A) \to \mathbb{R}$ be given by $V_A(p) := \max_{\sigma,\tau}\{\sigma \bar{A}(p)\tau | \sigma \bar{A}(p)\tau \leq \mathrm{Cav}(v_A)(p)\}$. For this example we have that $V_A(p) = v_A(p), \forall p \in \triangle(K_A)$, which can be checked by computation. Let $(\beta_{A,s})_{s\in\mathbb{N}^*}, (\beta_{B,s})_{s\in\mathbb{N}^*}$ be the martingales of expected payoffs associated to this equilibrium.

For each $s \in \mathbb{N}^*$, $\beta_{A,s} \leq V_A(p_{sA}) + Z_s$ a.s., where $(Z_s)_{s\in\mathbb{N}^*}$ is a (a.s.) nonnegative, bounded sequence that converges (a.s.) to 0.[20] Therefore, $\beta_{A,s} \leq v_A(p_{sA}) + Z_s$ a.s.. Letting $s \to \infty$, we have, by the Martingale

[20]See Lemma 6.11, in the Appendix, for a proof of this result.

FIGURE 6. Graphs of $\mathrm{Cav}(v_B)$ and $v_B$




Convergence Theorem, $\beta_{A,s} \to \beta_{A,\infty}, p_{sA} \to p_A^\infty$ which then implies by the Dominated Convergence Theorem $\mathbb{E}[\beta_{A,\infty}] \le \mathbb{E}[v_A(p_A^\infty)]$. Since by assumption $\mathbb{E}[\beta_{A,\infty}] = \mathrm{Cav}(v_A)(p_A^0)$, it follows that $\mathbb{E}[\beta_{A,\infty}] = \mathbb{E}[v_A(p_{\infty A})]$, which implies that the distribution of $p_{\infty A}$ is concentrated at the boundary of $\Delta(K_A)$. Since $v_B$ is strictly concave and by Proposition 5.5 $(v_B(p_{sB}))_{s\in\mathbb{N}^*}$ is a martingale, it follows that $p_{sB} = p_B^0$ a.s., $\forall s \in \mathbb{N}^*$. Hence, we have that for any history $h_\infty$ outside a set of $\mathbb{P}_{\sigma,\tau_A,\tau_B,p}$-measure zero, the bi-stochastic matrix representation of $p^\infty(h_\infty)$ has either the first or the second row filled with zeros (recall that an entry $p_{ij}^0$ represents the probability of states $i$ and $j$ in game $G_A(p_A^0)$ and $G_B(p_B^0)$, respectively), i.e., $p^\infty(h_\infty)$ is either:

$$\begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

Now the process of posteriors is a martingale, which implies that the expectation of $p^\infty$ is $p^0$. This implies that the following equation has a solution in $\lambda \in [0,1]$:

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}.$$

But this equation has no solution for $\lambda \in [0,1]$, which finally implies a contradiction. Hence, there is no equilibrium paying ex-ante to the informed player $\mathrm{Cav}(v_A)(p_A^0) + \mathrm{Cav}(v_B)(p_B^0)$. The arguments above indeed give us more: recall that we had $\beta_{A,\infty} \le V_A(p_A^\infty) = v_A(p_A^\infty)$ a.s. and since $\beta_{B,\infty} \le \mathrm{Cav}(v_B)(p_B^\infty) = v_B(p_B^\infty)$ a.s. by Lemma 5.4, these imply that $\beta_{A,\infty} + \beta_{B,\infty} \le v_A(p_A^\infty) + v_B(p_B^\infty)$ a.s. and therefore $\mathbb{E}[\beta_{A,\infty} + \beta_{B,\infty}] \le \mathbb{E}[v_A(p_A^\infty) + v_B(p_B^\infty)] \le \mathbb{E}[\mathrm{Cav}(\bar{h})(p^\infty)] \le \mathrm{Cav}(\bar{h})(p^0)$, where the second inequality follows by definition of $\mathrm{Cav}(\bar{h})$ and the last inequality is given by Jensen's inequality. The number $\mathrm{Cav}(\bar{h})(p^0)$ is the lowest possible ex-ante equilibrium payoff to the informed player, by Lemma 4.8. This implies that every uniform equilibrium of the example pays $\mathrm{Cav}(\bar{h})(p^0)$ to the informed player.

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## 6. Appendix

**Approachability Strategies.** In this section we recall the definition of approachability strategy, due to Blackwell et al. (1956). This section is taken from Sorin (2002) and adapted to our setting. Fix $p \in \Delta(K_A \times K_B)$.

Let $C$ be a $|I_A \times I_B| \times |J_A \times J_B|$-matrix with coefficient in $\mathbb{R}^{\text{supp}(p)}$, where

$$C_{i_A,j_A,j_A,j_B} = (C^{k_A,k_B}_{i_A,i_B,j_A,j_B})_{(k_A,k_B)\in\text{supp}(p)} = (a^{k_A}_{i_A,j_A} + b^{k_B}_{i_B,j_B})_{(k_A,k_B)\in\text{supp}(p)},$$

where $a^{k_A}_{i_A,j_A} \in \mathbb{R}$ and $b^{k_B}_{i_B,j_B} \in \mathbb{R}$. We define a *vector payoff zero-sum game*: at stage $n$, player 1 (respect. player 2) chooses a move $(i^n_A, i^n_B)$(respect. $(j^n_A, j^n_B)$). The corresponding vector payoff $g_n = C_{i^n_A,j^n_A,j^n_A,j^n_B}$ is announced. Denote by $h_n$ the sequence of vector payoffs at stage $n$. This is the information available to both players up to stage $n$. Let $\overline{g}_n = \frac{1}{n}\sum_{m=1}^{n} g_m$ be the vector of average payoffs up to stage $n$. Let $||C|| = \max_{i_A,i_B,j_A,j_B,k_A,k_B} |C^{k_A,k_B}_{i_A,i_B,j_A,j_B}|$.

**Definition 6.1.** A set $P \subset \mathbb{R}^{\text{supp}(p)}$ is approachable by player 2 if for any $\epsilon > 0$ there exists strategy $\tau$ of player 2 and $N \in \mathbb{N}^*$ such that for any strategy $\sigma$ of player 1 and $n \geq N$:

$$\mathbb{E}_{\sigma,\tau}[d_n] \leq \epsilon,$$

where $d_n$ is the euclidean distance $d(\overline{g}_n, P)$.

Let

$$C\tau = \text{co}\{\sum_{j_A,j_B} C_{i_A,i_B,j_A,j_B}\tau_{j_A,j_B} | (i_A,i_B) \in I_A \times I_B\},$$

where $C_{i_A,i_B,j_A,j_B}\tau_{j_A,j_B} := (C^{k_A,k_B}_{i_A,i_B,j_A,j_B}\tau_{j_A,j_B})_{(k_A,k_B)\in\text{supp}(p)}$.

**Definition 6.2.** A closed set $P \subset \mathbb{R}^{\text{supp}(p)}$ is a $B$-set for player 2 if: for any $z \notin P$ there exists a closest point $y = y(z)$ in $P$ to $z$ and a mixed move $\tau = \tau(z) \in \Delta(J_A \times J_B)$, such that the hyperplane through $y$ orthogonal to the segment $[y, z]$ separates $z$ from $C\tau$.

**Theorem 6.3.** *Let $P$ be a $B$-set for player 2. Then $P$ is approachable by that player. More precisely with a strategy satisfying $\tau(h_{n+1}) = \tau(\overline{g}_n)$, whenever $\overline{g}_n \notin P$, one has:*

$$\mathbb{E}_{\sigma,\tau}[d_n] \leq \frac{2||C||}{\sqrt{n}}, \forall\sigma$$

*and $d_n$ converges $\mathbb{P}_{\sigma,\tau}$ a.s. to 0.*

*Proof.* See Theorem B1 in Sorin (2002). □

**Remark 6.4.** The strategy $\tau$ obtained in the statement of Theorem 6.3 above will be called an *approachability strategy.*

For $z \in \mathbb{R}^{\text{supp}(p)}$, let $M(z) := z - \mathbb{R}^{\text{supp}(p)}_{+}$.

**Theorem 6.5.** *Let $z \in Z_\infty = \{z \in \mathbb{R}^{\text{supp}(p)} | z \cdot q \geq \bar{h}(q), \forall q \in \Delta(\text{supp}(p))\}$. Then player 2 can approach $M(z)$. Also, the approachability strategy $\tau$ for player 2 can be assumed to satisfy $\tau(h_t) = \tau^A(h_t)\bigotimes\tau^B(h_t) \in \Delta(J_A)\bigotimes\Delta(J_B), \forall h_t \in H_t, t \in \mathbb{N}^*$.*

*Proof.* Approachability of $M(z)$ follows from Theorem 3.33 in Sorin (2002), where it is checked that $M(z)$ is a $B$-set. Fix $\overline{g}_n$ and let $\tau(\overline{g}_n)$ be the mixed move associated with the $B$-set $M(z)$ and $\overline{g}_n$ in Theorem 6.3. Let $\tau^A = \text{marg}_{J_A}\tau$ and $\tau^B = \text{marg}_{J_B}\tau$. We show $C\tau = C(\tau^A\bigotimes\tau^B)$: let $x \in C\tau$ such that $x := \sum_{(i_A,i_B)} \lambda^{i_A,i_B} \sum_{j_A,j_B} C_{i_A,i_B,j_A,j_B}\tau_{j_A,j_B}$ with $\sum_{i_A,i_B}\lambda^{i_A,i_B} = 1$ and $\lambda^{i_A,i_B} \geq 0$. Then,

$$\sum_{(i_A,i_B)} \lambda^{i_A,i_B} \sum_{j_A,j_B} C_{i_A,i_B,j_A,j_B}\tau_{j_A,j_B} = \sum_{(i_A,i_B)} \lambda^{i_A,i_B} \sum_{j_A,j_B} (a^{k_A}_{i_A,j_A} + b^{k_B}_{i_B,j_B})_{(k_A,k_B)}\tau_{j_A,j_B} =$$
$$\sum_{i_A,i_B} \lambda^{i_A,i_B} [\sum_{j_A} a^{k_A}_{i_A,j_A}\tau^A_{j_A} + \sum_{j_B} b^{k_B}_{i_B,j_B}\tau^B_{j_B}]_{(k_A,k_B)} = \sum_{i_A,i_B}\lambda^{i_A,i_B}\sum_{j_A,j_B}(a^{k_A}_{i_A,j_A} + b^{k_B}_{i_B,j_B})_{(k_A,k_B)}\tau^A_{j_A}\tau^B_{j_B} =$$
$$\sum_{i_A,i_B}\lambda^{i_A,i_B}\sum_{j_A,j_B} C_{i_A,i_B,j_A,j_B}\tau^A_{j_A}\tau^B_{j_B},$$

which implies that $x \in C(\tau^A \bigotimes \tau^B)$. Now, let $y := \sum_{(i_A,i_B)} \lambda^{i_A,i_B} \sum_{j_A,j_B} C_{i_A,i_B,j_A,j_B} \tau^A_{j_A} \tau^B_{j_B} \in C(\tau^A \bigotimes \tau^B)$, with $\sum_{(i_A,i_B)} \lambda^{i_A,i_B} = 1$ and $\lambda^{i_A,i_B} \geq 0$. Using the the same equalities above, it implies $y \in C\tau$. So, if player 2 uses $\tau_t(h_t) := \tau^A(h_t) \bigotimes \tau^B(h_t)$ after each $h_t$ , $\tau$ is an approachability strategy. □

**Remark 6.6.** Note that the marginal $\tau^A$ (respectively $\tau^B$) in the proof above is not necessarily the approachability strategy of $G_A(p_A)$ (respectively $G_B(p_B)$). This is the case if and only if the interval $I(p)$ is degenerate, which as examples in this paper show, is not always true.

**Proof of Lemma 3.6.** Let $(S, x, \gamma)$ be a safe independent joint-plan. We construct an equilibrium $(\sigma, \tau_A, \tau_B)$ in $\mathcal{G}(p)$ inducing the required joint-plan vector of payoffs. The proof is exactly analogous to Proposition 1 in Sorin (1983). We indicate the steps of the construction. If $(k_A, k_B) \in \text{supp}(p)$ realizes, player 1 uses a *state-dependent lottery* (see Lemma 1 in Sorin (1983)), using finitely many stages of his play to induce one of the posteriors $(p(s))_{s\in S}$. For each $s \in S$, Lemma 2 of Sorin (1983) implies there exists $h^s_{A,\infty} = ((i^n_A, j^n_A))_{n\geq 1}$ such that for each $(i_A, j_A) \in I_A \times I_B$

$$\frac{1}{n}|\{m|1 \leq m \leq n, (i^m_A, j^m_B) = (i_A, j_A)\}| \to \gamma^s_A(i_A, j_A),$$

as $n \to +\infty$.

Assume the state-dependent lottery of player 1 draws signal $s \in S$. Then player 1 will play according to $h^s_{A,\infty}$, as long as player 2 plays according to $h^s_{A,\infty}$. The asymptotic frequency induced by this deterministic play is $\gamma^s_A$. The expected payoff to player 2 given that $s$ realizes is then $-\beta_A(s)$. Similarly, players 1 and 3 will play a deterministic sequence of moves according to some $h^s_{B,\infty}$ such that the induced asymptotic frequency is $\gamma^s_B$. The expected payoff to player 3 given that $s$ realizes is then $-\beta_B(s)$. The payoff to player 1 from following the deterministic sequence of moves in both games is $\alpha^{k_A,k_B}(s) = \alpha^{k_A,k_B}$, $\forall s \in S$ that realize with positive probability, by condition (2). If any player deviates from the deterministic sequence of moves they are supposed to play after $s$ realizes, the deviation is detectable. If player 2 deviates, player 1 will punish him by playing the optimal strategy of the zero-sum game $G_A(p_A(s))$. If player 3 deviates, player 1 will punish him by playing the optimal strategy of the zero-sum game $G_B(p_B(s))$. If player 1 deviates from his deterministic sequence in either game he is playing, the deviation is detectable by players 2 and 3 and players 2 and 3 will play the approachability strategy given by Theorem 6.5. During signaling stages, players 2 and 3 can play anything, but if player 1 makes a detectable deviation during signaling stages – by not using a signal $s$ – then players 2 and 3 will also play the approachability strategy given by Theorem 6.5. We show that the strategies defined form a uniform equilibrium: first, condition (2) of Theorem 3.6 prevents any undetectable deviation of player 1 at signaling stages from being profitable (see Mertens, Sorin and Zamir (2015), proof of Proposition IX 1.1). After signal $s$ realizes, if player 2 deviates, then player 1 plays the optimal strategy of $G_A(p_A(s))$. Since $\beta_A(s) \leq \text{Cav}(v_A)(p(s)_A)$, it implies that the deviation is not profitable for player 2. The same reasoning applies for a deviation of player 3. The inequality $\alpha \cdot q \geq \overline{h}(q), \forall q \in \Delta(\text{supp}(p))$ of condition (3), shows that a deviation is not profitable for player 1, because $M(\alpha)$ is approachable by players 2 and 3, by Theorem 6.5.

Condition (1) of the uniform equilibrium definition is immediately satisfied, because the payoffs to each player given by the strategies defined above converge. Also, condition (2) of the definition follows immediately from the fact that deviations are punished with approachability strategies, in case player 1 deviates, and optimal strategies of the zero-sum games, in case players 2 or 3 deviate.

**Proof of Lemma 3.7.** We first state Theorem 6.7 and Lemma 6.8 below, which are the main tools for the proof of Lemma 3.7.

**Theorem 6.7.** *[Simon, Spież and Toruńczyk (1995)] Let $K$ be a finite set and $p^0 \in int \triangle(K)$. Let $a : \triangle(K) \to \mathbb{R}$ and $h : \triangle(I) \times \triangle(K) \to \mathbb{R}^{|K|}$ be continuous functions such that:*

1. *The function $h$ is affine with respect to the variable $\sigma \in \triangle(I)$, for all $p \in \triangle(K)$.*
2. *For all $p, q \in \triangle(K)$, there is $\sigma \in \triangle(I)$ such that $h(\sigma, p) \cdot q \geq a(q)$.*
   *Therefore, there exists a set $P_0 \subset \triangle(K)$ of cardinality $\leq |K|$ and vectors $\sigma_p \in \triangle(I)$ (with $p \in P_0$) and $\phi \in \mathbb{R}^{|K|}$ such that:*
3. *$\phi \cdot q \geq a(q)$ for all $q \in \triangle(K)$.*
4. *$p^0 \in coP_0$.*
5. *For all $p \in P_0$, $k \in K$ we have $\phi^k \geq h^k(\sigma_p, p)$, with equality occuring in place of inequality whenever $p^k > 0$.*

We will also make use of the following simple version of a lemma in Simon, Spież and Toruńczyk (1995):

**Lemma 6.8.** *For every $\epsilon > 0$ there exists a continuous map $g_A : \triangle(K_A) \to \triangle(J_A)$ such that $\sigma_A \bar{A}(p)(g_A(p))^T \leq v_A(p) + \epsilon$, for all $(\sigma, p) \in \triangle(I_A) \times \triangle(K_A)$.*

*Proof of Lemma 3.7.* Assume first $p^0 \in \text{int}\Delta(K_A \times K_B)$. Given $\epsilon > 0$, applying Lemma 6.8 we have that $\sigma_A \bar{A}(p_A)(g_A(p_A))^T \leq v_A(p_A) + \epsilon$ and $\sigma_B \bar{B}(p_B)(g_B(p_B))^T \leq v_B(p_B) + \epsilon$, for all $(\sigma_A, p_A) \in \triangle(I_A) \times \triangle(K_A)$ and $(\sigma_B, p_B) \in \triangle(I_B) \times \triangle(K_B)$. Define $h(\sigma, p) = ((m_A\sigma)A^{k_A}(g_A(p_A))^T + (m_B\sigma)B^{k_B}(g_B(p_B))^T)_{(k_A,k_B) \in K_A \times K_B}$, where $m_A\sigma := \text{marg}_{I_A}\sigma$ and $m_B\sigma := \text{marg}_{I_B}\sigma$. Since the marginal operator is affine, the function $h$ is affine on $\sigma$. It is also continuous. Now, given $p, q \in \triangle(K_A \times K_B)$, let $\bar{\sigma}^q_A$ be the optimal strategy of the informed player in the one-shot zero-sum game with matrix $\bar{A}(q_A)$ and let $\bar{\sigma}^q_B$ be the optimal strategy of the informed player in the one-shot zero-sum game with matrix $\bar{B}(q_B)$. Define $\tilde{\sigma} := \bar{\sigma}^q_A \bigotimes \bar{\sigma}^q_B \in \Delta(I_A \times I_B)$. Then we have that $h(\tilde{\sigma}, p) \cdot q \geq a(q) := \bar{h}(q) = v_A(q_A) + v_B(q_B)$. Applying Theorem 6.7, we have that there exists $P_0 \subset \triangle(K_A \times K_B)$ with cardinality $\leq |K_A \times K_B|$ , $(\sigma_p)_{p \in P_0}$ and $\phi \in \mathbb{R}^{|K_A \times K_B|}$ satisfying (3), (4) and (5). From (3) and (4) we have that there exists a nonnegative collection $(\lambda_p)_{p \in P_0}$ and a vector $\phi$ such that $\sum_{p \in P_0} \lambda_p p = p^0$ and $\sum_{p \in P_0} \lambda_p = 1$, $\phi \cdot q \geq a(q)$ and $(m_A\sigma_p)\bar{A}(p_A)(g_A(p_A))^T \leq v_A(p_A) + \epsilon$ and $(m_B\sigma_p)\bar{B}(p_B)(g_B(p_B))^T \leq v_B(p_B) + \epsilon$ for $\sigma_p \in \triangle(I_A \times I_B)$ and $p \in P_0$.

Notice that the solutions given by the application of Theorem 6.7 are all indexed by $\epsilon > 0$. For each $n \in \mathbb{N}^*$, we can therefore consider $P_0^n = \{p_s^n\}_{s \in S_n} \subset \Delta(K_A \times K_B)$ such that $|P_0^n| \leq |K_A \times K_B|$, $(\sigma_{p^n})_{p^n \in P_0^n}$, $(m_B\sigma_{p_s^n})$, $g_B(p_{sB}^n)$, $(m_A\sigma_{p_s^n})$, $g_A(p_{sA}^n)$, $(\lambda_{p_s^n})_{s \in S_n}$ and $\phi_n$ satisfy $(m_B\sigma_{p_s^n})\bar{B}(p_{sB}^n)(g_B(p_{sB}^n))^T \leq v_B(p_{sB}^n) + 1/n$ and $(m_A\sigma_{p_s^n})\bar{A}(p_{sA}^n)(g_A(p_{sA}^n))^T \leq v_A(p_{sA}^n) + 1/n$ such that $\sum_s \lambda_{p_s^n} p_s^n = p^0$ and $\sum_s \lambda_{p_s^n} = 1$; also, $\phi_n$ such that $\phi_n \cdot q \geq a(q), \forall q$ with (5) being satisfied.

Passing to a subsequence if necessary, consider the (Hausdorff) limit $P_0$ of the sequence $P_0^n$. Notice that $P_0$ has finite cardinality (less than $|K_A \times K_B|$). We can also consider limits of the associated solutions, since they all lie in compact sets.[21] Therefore, consider $S$ a finite set, $P_0 = \{p_s\}_{s \in S}$,

[21]Property (5) of Theorem 6.7 guarantees then that the sequence of vectors $(\phi_n)_{n \in \mathbb{N}^*}$ is bounded, so it will also have an accumulation point.

$(\sigma_{p_s})_{s\in S}, (g_A(p_{sA}))_{s\in S}, (g_B(p_{sB}))_{s\in S}$, $(\lambda_{p_s})_{s\in S}$ and $\phi$ to be the limits of those sequences. It is straightforward to check that the limit of the sequences of solutions satisfy (3), (4) and (5). The joint-plan is now defined as follows: consider as contracts $\gamma_A^s = m_A\sigma_{p_s}\bigotimes g_A(p_{sA})$, $\gamma_B^s = m_B\sigma_{p_s}\bigotimes g_B(p_{sB})$.

Let $\tau_{p_s}^A = (g_A(p_{sA}))^T$ and $\tau_{p_s}^B = (g_B(p_{sB}))^T$. By construction, $\max_\sigma \sigma\bar{A}(p_{sA})\tau_{p_s}^A = v_A(p_{sA})$ and $\max_\sigma \sigma\bar{B}(p_{sB})\tau_{p_s}^B = v_B(p_{sB})$. Therefore it implies that $(m_A\sigma_{p_s})\bar{A}(p_{sA})\tau_{p_s}^A \leq v_A(p_{sA})$ and $(m_B\sigma_{p_s})\bar{B}(p_{sB})\tau_{p_s}^B \leq v_B(p_{sB})$, for each $s\in S$. This implies in particular (1) of Lemma 3.6 is satisfied. Condition (3) of Lemma 3.6 follows directly from the fact that condition (5) of Theorem 6.7 above is satisfied by $\phi$. We can now define the signaling strategy $x$: let $S\subset H_{t_0}^1$ with $|H_{t_0}^1|\geq |K_A\times K_B|$, for $t_0\in\mathbb{N}^*$. Define $x^{k_A,k_B}(s) = \lambda_{p_s}\frac{p_s^{k_A,k_B}}{p^{0,k_A,k_B}}$. This signaling strategy satisfies condition (2) of Lemma 3.6 and induces the appropriate posteriors: in case $s\in S$ is observed, the uninformed players will update their priors according to Bayes rule to posterios $p_s$.

We now calculate the ex-ante payoffs of player 1 obtained from the joint-plan just defined. Using condition (5) of Theorem 6.7, we have that $\sum_{p_s}\lambda_{p_s}(h(\sigma_{p_s},p_s)\cdot p_s) = \phi\cdot p^0 = \text{Cav}(\bar{h})(p^0)$. Rewriting the expression for $h$, $\sum_{p_s}\lambda_{p_s}(m_A\sigma_{p_s}\bar{A}(p_{s_A})\tau_{p_s}^A + m_B\sigma_{p_s}\bar{B}(p_{s_B})\tau_{p_s}^B) = \text{Cav}(\bar{h})(p^0)$. This proves the result for $p^0\in\text{int}\Delta(K_A\times K_B)$. If now $p^0\in\partial\Delta(K_A\times K_B)$, then consider the model $\mathcal{G}(\bar{p})$ defined for $\bar{p}\in\text{int}\Delta(\text{supp}(p^0))$, where $\bar{p}^{k_A,k_B} = p^{0,k_A,k_B}, \forall(k_A,k_B)\in\text{supp}(p^0)$ and apply the result proved to this case. □

**Proof of Proposition 4.4.** Suppose the zero-sum games $G_A(p_A)$ and $G_B(p_B)$ are not perfectly aligned. This means that for each optimal strategy of game $G_{A+B}(p)$, there is one zero-sum game, say $G_A(p_A)$, and one posterior $p_s\in\Delta(K_A\times K_B)$ induced by this optimal strategy for which $v_A(p_{sA}) < \text{Cav}(v_A)(p_{sA})$. Since for any posterior $p_s$ induced by the optimal strategy of $G_{A+B}(p)$ we have that $v_B(p_{sB})\leq\text{Cav}(v_B)(p_{sB})$, using Jensen's inequality we have that $\text{Cav}(\bar{h})(p) < \text{Cav}(v_A)(p_A)+\text{Cav}(v_B)(p_B)$. The converse is straightforward.

**Proof of Lemma 4.6.** We first construct a jointly controlled lottery with outcome set $\mathcal{O} = \{s_1,...,s_n\}$ and probability vector $(\lambda_i)_{i=1...,n}$, $\lambda_i > 0$, for $n\in\mathbb{N}^*$. In Aumann, Maschler and Stearns (1995) p.274, a jointly controlled lottery is only defined in two outcomes with any probabilities $\alpha$ and $1-\alpha$, so this generalization is necessary. First, consider a jointly controlled lottery on two outcomes $\{s_1\}$ and $\{s_2,....,s_n\}$ with probabilities $\lambda_1$ and $(1-\lambda_1)$, respectively. If the outcome is $\{s_2,....,s_n\}$, start another jointly controlled lottery on outcomes $\{s_2\}$ and $\{s_3,....,s_n\}$ with probabilities $\frac{\lambda_2}{1-\lambda_1}$ and $(1-\frac{\lambda_2}{1-\lambda_1})$ and so on. This procedure defines a jointly controlled lottery with outcomes $\mathcal{O}$ with the desired probabilities.

We start by proving (1). The proof of (2) is analogous. In case $k_A\in K_A$ is realized by Nature, players 1 and 2 will run a jointly controlled lottery with outcomes $\mathcal{O}_A$ and probabilities $(\lambda_s^A)_{s\in\mathcal{O}_A}$. In case outcome $s_A\in\mathcal{O}_A$ realizes, players 1 and 2 will play a deterministic sequence of moves $h_{A,\infty}^s$ such that the asymptotic frequency induced by such sequence is $\gamma_s^A$ (see proof of Lemma 3.6). This procedure reveals no information about the states chosen by Nature. The expected vector of payoffs to player 1 of such plan is $\alpha^A$, which, by assumptions (i) of Lemma 4.6 and Theorem 3.33 in Sorin (2002) implies that: letting $\alpha_*^A := (\alpha^{k_A})_{k_A\in\text{supp}(p_A)}$ and $K_A^* = \text{supp}(p_A)$, then $\alpha_*^A - \mathbb{R}_+^{K_A^*}$ is approachable by player 2. Also, the expected payoff to player 2 of this path of play is $-\beta_A$ and $\beta_A$ satisfies condition (ii) of Lemma 4.6. Therefore, in case any player deviates from the path he is supposed to play, the other player will observe the deviation and play the optimal strategy of the game $G_A(p_A)$, which renders any deviation not profitable. A known property of jointly controlled lotteries is that during lottery stages no undetectable deviation is profitable (see Aumann, Maschler and Stearns (1995)). If any player makes a detectable deviation then the other player plays the optimal strategy of the zero-sum game $G_A(p_A)$. Conditions (i) and (ii) then guarantee that the profile of strategies just defined for players 1 and 2 is a uniform equilibrium, since the optimal strategy of the informed player guarantees the uninformed player does not get more than $-\beta_A$ and the optimal strategy of the uninformed player guarantees the informed player does not get more than $\alpha^{k_A}$ for each type $k_A$.

We now prove (3): in case $(k_A, k_B) \in K_A \times K_B$ is realized by Nature, players 1 and 2 will run a jointly controlled lottery with outcomes $\mathcal{O}_A$ and probabilities $(\lambda_s^A)_{s\in\mathcal{O}_A}$. In case outcome $s_A \in \mathcal{O}_A$ realizes, players 1 and 2 will play a deterministic sequence of moves $h^s_{A,\infty}$ such that the asymptotic frequency induced by such sequence is $\gamma_s^A$. This procedure reveals no information about the states chosen by Nature. Also, players 1 and 3 will play a jointly controlled lottery with outcomes $\mathcal{O}_\mathcal{B}$ and probabilities $(\lambda_s^B)_{s\in\mathcal{O}_B}$ and if $s_B \in \mathcal{O}_B$ realizes, players 1 and 3 play a deterministic sequence of moves with asymptotic frequency $\gamma_s^B$. Notice that the payoff to player 1 according to this plan is $\alpha^{k_A} + \alpha^{k_B}$. Conditions (1.i) and (2.i) of Lemma 4.6 imply that for each $q \in \Delta(\text{supp}(p))$, $(\alpha^{k_A} + \alpha^{k_B})_{(k_A,k_B)\in\text{supp}(p)} \cdot q \geq \overline{h}(q)$, which then implies that $M(\alpha)$ is approachable by players 2 and 3, where $\alpha := (\alpha^{k_A} + \alpha^{k_B})_{(k_A,k_B)\in\text{supp}(p)}$. If player 2 makes a detectable deviation, then player 1 punishes him with the optimal strategy of $G_A(p_A)$ and similarly if player 3 makes a deviation. If player 1 makes a detectable deviation, players 2 and 3 play the approachability strategy of Theorem 6.5 and approach $M(\alpha)$. The profile of strategies defined is a uniform equilibrium: first notice that payoffs converge, as required by condition (1) of the definition of uniform equilibrium. Second, conditions (1.ii) and (2.ii) of Lemma 4.6 guarantee that no profitable deviations can be made by the uninformed players and since $M(\alpha)$ is approachable, no profitable deviation can be made by the informed player. Since any deviation is punished with an optimal strategy, this implies condition (2) of the definition of uniform equilibrium is satisfied.

**Proof of Lemma 4.7.** We first prove two technical results:

**Lemma 6.9.** *Let $p \in \Delta(K_A)$ and $Cav(v_A)(p) > v_A(p)$.*[22] *Let $\phi \in \mathbb{R}^{supp(p)}$ be such that:*

(1) $\phi \cdot q \geq v_A(q), \forall q \in \Delta(supp(p))$.
(2) $\exists \overline{p} \in int\Delta(supp(p))$ *such that* $\phi \cdot \overline{p} = Cav(v_A)(\overline{p}) = v_A(\overline{p})$.
(3) $\phi \cdot p = Cav(v_A)(p)$.

*Then there exists $\{\lambda_k | \lambda_k \geq 0$ with $k = 1, ..., |supp(p)|$ and $\sum_k \lambda_k = 1\}$, $\{\sigma^k | k = 1, ..., |supp(p)|\} \subset \Delta(I_A)$ and $\{\tau^k | k = 1, ..., |supp(p)|\} \subset \Delta(J_A)$ such that:*

$$\phi = \sum_k \lambda_k (\sigma^k A^{k_A} (\tau^k)^T)_{k_A \in supp(p)}.$$

*Proof.* We assume without loss of generality that $\text{supp}(p) = K_A$. This is only to simplify notation and the proof is the exact same otherwise. Let $e_i^{K_A-1} = (0, .., 1, ..., 0) \in \mathbb{R}^{K_A-1}$ with one in the $i$-th position. Analogously, let $e_i^{K_A} = (0, .., 1, ..., 0) \in \mathbb{R}^{K_A}$. Define $T : \mathbb{R}^{K_A-1} \to \mathbb{R}^{K_A}$ as the affine transformation that maps $e_i^{K_A-1} \mapsto e_{i+1}^{K_A}$ and $0 \mapsto e_1^{K_A}$, for $i \in \{1, 2, ..., K_A - 1\}$. Since $T$ is affine, $Tx = Sx + e_1^{K_A}$, where $S$ is a linear transformation – we will also denote by $S$ the matrix representation of $S$ according to the canonical basis. Notice that $T$ is injective. Let now $P = \text{co}\{\{e_1^{K_A-1}, ..., e_{K_A-1}^{K_A-1}\} \cup \{0\}\}$ and define the restriction $T^\star := T|_P$ and $f := v_A \circ T^\star : P \to \mathbb{R}$. Since $v_A$ is Lipschitz [23] and $T^\star$ is affine, $f$ is also Lipschitz.

[22]Note that $\text{Cav}(v_A)(p) > v_A(p)$ implies $|\text{supp}(p)| \geq 2$.
[23]See Sorin (2002), Proposition 2.1.

Let $V \subset \text{int}P$ be an open neighborhood of $x_0 \in \text{int}P$ such that $T^\star x_0 = \overline{p}$. It follows by Radamacher Theorem[24] that $f$ is almost everywhere differentiable in $V$. Let $x_0+h, x_0+h+v \in V$ with $x_0+h$ a point of differentiability of $f$. Then we can write: $f(x_0+h+v) = f(x_0+h) + \nabla f(x_0+h)\cdot v + o(||v||)$.

Let $f_\tau^A(q) := \max_\sigma \sigma \bar{A}(q)(\tau)^T, q \in \triangle(K_A)$ and define $g_\tau := f_\tau^A \circ T^\star$. By definition, denoting by $\tau_x$ an optimal strategy of the minimizer at the zero-sum game with matrix $\bar{A}(T^\star(x))$, we have that:

$$g_{\tau_{x_0+h}}(x_0+h+v) \geq v_A(T^\star(x_0+h+v)) = v_A(T^\star(x_0+h)) + \nabla f(x_0+h)\cdot v + o(||v||).$$

But $v_A(T^\star(x_0+h)) = g_{\tau_{x_0+h}}(x_0+h)$. This gives $g_{\tau_{x_0+h}}(x_0+h+v) \geq g_{\tau_{x_0+h}}(x_0+h)+\nabla f(x_0+h)\cdot v + o(||v||)$. Since $g_{\tau_{x_0+h}}$ is a convex function, all its directional derivatives exist and we have that $g_{\tau_{x_0+h}}(x_0+h+v) \geq g_{\tau_{x_0+h}}(x_0+h)+\nabla f(x_0+h)\cdot v$. This implies that $\nabla f(x_0+h)$ is a subgradient of the convex function $g_{\tau_{x_0+h}}$ at the point $x_0+h$.

Note now that $g_{\tau_{x_0+h}}$ is piecewise linear and its epigraph $\mathcal{B}$ is therefore a polyhedral set. This implies every maximal proper face of $\mathcal{B}$ is contained in a hyperplane $H_i$ of $\mathbb{R}^{K_A-1}\times\mathbb{R}$ given by

$$H_i = \{(x,t) \in \mathbb{R}^{K_A-1}\times\mathbb{R} | t = (s_i A^{k_A}\tau_{x_0+h})_{k_A\in K_A}\cdot T^\star x\},$$

where $s_i \in I_A$. Define $\overline{y} = (x_0+h, f(x_0+h))$. Notice that $(x_0+h, f(x_0+h)) = (x_0+h, g_{\tau_{x_0+h}}(x_0+h))$. Let $I_0$ be the set of all pure strategies $s_i$ corresponding to maximal proper faces of $\mathcal{B}$ that contain $\overline{y}$. The graph of the affine function $r(v) = f(x_0+h)+\nabla f(x_0+h)\cdot v$ is a hyperplane of $\mathbb{R}^{K_A-1}\times\mathbb{R}$ that supports $\mathcal{B}$ at $\overline{y} = (x_0+h, f(x_0+h))$. Now, an application of Farkas' lemma[25] will give us that there exists $\sigma_{x_0+h} \in \triangle(I_0)$ such that $\nabla f(x_0+h) = \sum_{s\in I_0}\sigma_{x_0+h}(s)(sA^{k_A}(\tau_{x_0+h})^T)_{k_A\in K_A}S$ and because $H_i$ supports $\mathcal{B}$ at $\bar{y}$, it implies that

$$\sum_{s\in I_0}\sigma_{x_0+h}(s)(sA^{k_A}(\tau_{x_0+h})^T)_{k_A\in K_A}\cdot T^\star(x_0+h) = f(x_0+h).$$

Let $\phi$ be the row vector from the statement of the Lemma. Note that $\phi S$ is a supergradient of $f$ at $x_0$, therefore it is in the generalized (Clarke) superdifferential $\partial f(x_0) := \text{co}\{\lim \nabla f(x+h_i), \text{ as } h_i \to 0 \text{ with } i\to\infty\}$[26]. Since $\partial f(x_0)$ is convex, Carathéodory Theorem allows us to write $\phi S$ as a convex combination of $|K_A|$ points $\mathcal{N}(x_0) := \{d_1,...,d_{|K_A|}\} \subset \{\lim\nabla f(x+h_i), \text{ as } h_i \to 0 \text{ as } i\to\infty\} : \sum_{k=1}^{|K_A|}\lambda_k d_k = \phi S$. Therefore, for each $k$, there exists a sequence $\{h_i^k\}_{i\in\mathbb{N}^*}$ such that $\nabla f(x_0+h_i^k) = (\sigma_{x_0+h_i^k}A^{k_A}\tau_{x_0+h_i^k})_{k_A\in K_A}S$ and $(\sigma_{x_0+h_i^k}A^{k_A}\tau_{x_0+h_i^k})_{k_A\in K_A}\cdot T^*(x_0+h_i^k) = f(x_0+h_i^k)$ such that $\nabla f(x_0+h_i^k) \to d_k$ as $i\to\infty$. Now, passing to a subsequence if necessary, we can assume that $\tau_{x_0+h_i^k} \to \tau_{x_0}^k \in \triangle(J_A)$ and $\sigma_{x_0+h_i^k} \to \sigma_{x_0}^k \in \triangle(I_A)$, as $i\to+\infty$. So $(\sigma_{x_0}^k A^{k_A}\tau_{x_0}^k)_{k_A\in K_A}S = d_k$ and $(\sigma_{x_0}^k A^{k_A}\tau_{x_0}^k)_{k_A\in K_A}\cdot\bar{p} = f(x_0) = v_A(\bar{p})$. This implies that $\phi S = \sum_k \lambda_k(\sigma_{x_0}^k A^{k_A}\tau_{x_0}^k)_{k_A\in K_A}S$ and $\sum_k \lambda_k(\sigma_{x_0}^k A^{k_A}\tau_{x_0}^k)_{k_A\in K_A}\cdot\bar{p} = f(x_0) = v_A(\bar{p})$. Now, because $S$ is full rank and $\bar{p}$ is in $\text{int}\Delta(K_A)$, the equalities $\sum_k \lambda_k(\sigma_{x_0}^k A^{k_A}\tau_{x_0}^k)_{k_A\in K_A}S = \phi S$ and $\phi\cdot\bar{p} = \sum_k \lambda_k(\sigma_{x_0}^k A^{k_A}\tau_{x_0}^k)_{k_A\in K_A}\cdot\bar{p}$ imply that $\phi = \sum_k \lambda_k(\sigma_{x_0}^k A^{k_A}\tau_{x_0}^k)_{k_A\in K_A}$. □

**Lemma 6.10.** *Let $p\in\Delta(K_A)$ with $G_A(p)$ locally nonrevealing at $p$ and $Cav(v_A)(p) > v_A(p)$. There exists $\phi\in\mathbb{R}^{supp(p)}$ such that:*

(1) $\phi\cdot q \geq v_A(q), \forall q\in\Delta(supp(p))$.

[24]Radamacher Theorem: If $U$ is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is Lipschitz continuous, then $f$ is differentiable (Lebesgue) almost everywhere in $U$.

[25]See Rockafellar (2015).

[26]See Clarke (1975).

(2) $\phi \cdot p = Cav(v_A)(p)$.

(3) $\exists \overline{p} \in int\Delta(supp(p))$ *such that* $\phi \cdot \overline{p} = Cav(v_A)(\overline{p}) = v_A(\overline{p})$.

*Proof.* Again we assume that $\text{supp}(p) = K_A$ only to simplify notation. By the theory of zero-sum games with lack of information on one-side, there exists a polytope $P_A \subset \Delta(K_A)$ containing $p$ in its relative interior and such that the vertices of $P_A$ are the posteriors induced by one optimal strategy of the informed player in game $G_A(p)$. By assumption that $G_A(p)$ is locally nonrevealing at $p$, one of these vertices, say $\overline{p}$, is such that $\overline{p} \in \text{int}\Delta(K_A)$, and $v_A(\overline{p}) = \text{Cav}(v_A)(\overline{p})$. Therefore, since $\text{Cav}(v_A)$ is affine at $P_A$, there exists $\phi_A \in \mathbb{R}^{K_A}$ such that $\text{Cav}(v_A)(q) = \phi_A \cdot q, \forall q \in P_A$ and $\phi_A \cdot q \geq v_A(q), \forall q \in \Delta(K_A)$. □

*Proof of Lemma 4.7.* Throughout the proof we will assume, only to simplify notation and without any loss of generality that, that $\text{supp}(p) = K_A$ and $\text{supp}(p) = K_B$.

**Case1.** Assume

$$v_A(p_A) < \text{Cav}(v_A)(p_A) \text{ and } v_B(p_B) = \text{Cav}(v_B)(p_B).$$

If $v_B(p_B) = \text{Cav}(v_B)(p_B)$, the optimal strategy of the informed player in the zero-sum game $G_B(p_B)$ is nonrevealing. Indeed, the behavior strategy $\sigma_B$ of the informed player in which he plays the optimal strategy $\sigma^B_{p_B}$ of the one-shot zero-sum game with matrix $\bar{B}(p_B)$ independently at each stage is the optimal strategy in $G_B(p_B)$ (see Sorin (2002)). The uninformed player 3 plays his (optimal) "approachability" strategy $\tau_B$ of $G_B(p_B)$. The *ex-ante* payoff obtained is $\text{Cav}(v_B)(p_B)$.

We now construct a nonrevealing loose joint-plan between players 1 and 2 in $G_A(p_A)$. First, by Lemma 6.10 there exists $\phi_A$ satisfying conditions (1), (2) and (3) of that Lemma. By Lemma 6.9, there exists $(\lambda_s)_{s\in\mathcal{O}_A}$, $(\sigma^s)_{s\in\mathcal{O}_A}$ and $(\tau^s)_{s\in\mathcal{O}_A}$ such that

$$\phi_A = \sum_{s\in\mathcal{O}_A} \lambda_s(\sigma^s A^{k_A}(\tau^s)^T)_{k_A\in K_A}.$$

Define $\gamma^A_s := \sigma^s \bigotimes \tau^s$ and let the nonrevealing loose joint-plan be given by $(\mathcal{O}_{\mathcal{A}}, (\lambda^A_s)_{s\in\mathcal{O}_A}, (\gamma^A_s)_{s\in\mathcal{O}_A})$.

Notice that $\alpha^A = (\alpha^{k_A})_{k_A\in K_A} = \sum_{s\in\mathcal{O}_A} \lambda^A_s(\sigma^s_{x_0} A^{k_A}\tau^s_{x_0})_{k_A\in K_A}$ and $\alpha^A \cdot q \geq v_A(q), \forall q \in \Delta(K_A)$, which is (i) in (1) of Lemma 4.6. Also $\beta_A = \text{Cav}(v_A)(p_A)$, which implies condition (ii) in (1) of Lemma 4.6. This implies there exists a (nonrevealing) uniform equilibrium $(\sigma_A, \tau_A)$ in $G_A(p_A)$. The profile of strategies where players 1 and 2 play $(\sigma_A, \tau_A)$ in game $G_A(p_A)$ and players 1 and 3 play the optimal strategies of $G_B(p_B)$ is a uniform equilibrium. The *ex-ante* equilibrium payoff of the informed player obtained in $\mathcal{G}(p)$ from this profile is $\alpha^A \cdot p_A + \text{Cav}(v_B)(p_B) = \phi_A \cdot p_A + \text{Cav}(v_B)(p_B) = \text{Cav}(v_A)(p_A) + \text{Cav}(v_B)(p_B)$.

**Case 2.** Assume

$$v_A(p_A) < \text{Cav}(v_A)(p_A) \text{ and } v_B(p_B) < \text{Cav}(v_B)(p_B).$$

By Lemma 6.10, there are two vectors $\phi_A$ and $\phi_B$ each satisfying (1), (2) and (3) of that Lemma. Applying Lemma 6.9, there exists $\{\lambda^A_s\}$ and $\{\lambda^B_r\}$, $\lambda^A_s \geq 0$ and $\lambda^B_r \geq 0$ with $s = 1, ..., |K_A|$ and $r = 1, ..., |K_B|$ and strategies $\sigma^s_A \in \Delta(I_A)$, $\sigma^r_B \in \Delta(I_B)$, $\tau^s_A \in \Delta(J_A)$ and $\tau^r_B \in \Delta(J_B)$ such that

$$\phi_A = \sum_s \lambda^A_s(\sigma^s_A A^{k_A}(\tau^s_A)^T)_{k_A\in K_A}$$

$$\phi_B = \sum_r \lambda^B_r(\sigma^r_B B^{k_B}(\tau^r_B)^T)_{k_B\in K_B}.$$

Consider now the nonrevealing loose joint-plans $(\mathcal{O}_A, (\lambda_s^A)_{s\in\mathcal{O}_A}, (\gamma_s^A)_{s\in\mathcal{O}_A})$ in $G_A(p_A)$ and $(\mathcal{O}_\mathcal{B}, (\lambda_r^B)_{r\in\mathcal{O}_B}, (\gamma_r^B)_{r\in\mathcal{O}_B})$ in $G_B(p_B)$ where $\mathcal{O}_A := \{1, ..., |K_A|\}$, $\mathcal{O}_B := \{1, ..., |K_B|\}$, $\gamma_s^A = \sigma_A^s \bigotimes \tau_A^s$ and $\gamma_r^B = \sigma_B^r \bigotimes \tau_B^r$. We check that conditions (1.i),(1.ii),(2.i) and (2.ii) of Lemma 4.6 are satisfied by the two defined nonrevealing loose joint-plans: first, (1.ii) is satisfied because

$$\beta_A = \alpha^A \cdot p_A = \sum_k \lambda_k^A (\sigma_A^k A^{k_A} \tau_A^k)_{k_A \in K_A} \cdot p_A = \phi_A \cdot p_A = \mathrm{Cav}(v_A)(p_A).$$

Second, (2.ii) is satisfied:

$$\beta_B = \alpha^B \cdot p_B = \sum_r \lambda_r^B (\sigma_B^r B^{k_B} \tau_B^r)_{k_B \in K_B} \cdot p_B = \phi_B \cdot p_B = \mathrm{Cav}(v_B)(p_B).$$

Conditions (1.i) and (2.i) are also satisfied: $\alpha^A \cdot q_A = \phi_A \cdot q_A \geq v_A(q_A), \forall q \in \Delta(K_A)$ and $\alpha^B \cdot q_B = \phi_B \cdot q_B \geq v_B(q_B), \forall q_B \in \Delta(K_B)$. Since (1.i),(1.ii),(2.i) and (2.ii) are satisfied, Lemma 4.6 implies that there exists a uniform equilibrium $(\sigma, \tau_A, \tau_B)$ with associated ex-ante payoffs to the informed player of $\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B)$.

**Case 3.** Assume:

$$v_A(p_A) = \mathrm{Cav}(v_A)(p_A) \;\text{ and }\; v_B(p_B) < \mathrm{Cav}(v_B)(p_B).$$

This case is symmetric to case 1.

**Case 4.** Assume

$$v_A(p_A) = \mathrm{Cav}(v_A)(p_A) \;\text{ and }\; v_B(p_B) = \mathrm{Cav}(v_B)(p_B).$$

This case is straightforward since the optimal strategy of the informed player in both games is nonrevealing. This is the case where $I(p)$ is degenerate, because optimal strategies are perfectly aligned (see Proposition 4.4). □

**Proof of Lemma 4.8.** We just have to show that the ex-ante equilibrium payoff set of the informed player is a subinterval. The fact that it contains the lower bound of $I(p)$ is a consequence of Lemma 3.7. Consider two ex-ante uniform equilibrium payoffs $\gamma_1$ and $\gamma_2$ for the informed player. Given $\alpha \in (0,1)$ we show that $\alpha\gamma_1 + (1-\alpha)\gamma_2$ is an ex-ante equilibrium payoff for the informed player in $\mathcal{G}(p)$. Since each of the uninformed players can play his optimal strategy in his repeated zero-sum game $G_A(p_A)$ or $G_B(p_B)$, it implies that

$$\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B) \geq \gamma_1$$

as well as

$$\mathrm{Cav}(v_A)(p_A) + \mathrm{Cav}(v_B)(p_B) \geq \gamma_2.$$

Now, since the informed player has an optimal strategy in the zero-sum game $G_{A+B}(p)$ that guarantees him $\mathrm{Cav}(\bar{h})(p)$, it implies that $\gamma_1 \geq \mathrm{Cav}(\bar{h})(p)$ as well as $\gamma_2 \geq \mathrm{Cav}(\bar{h})(p)$. Consider a jointly controlled lottery that implements the equilibrium profile associated with $\gamma_1$ with probability $\alpha$ and the equilibrium associated with $\gamma_2$ with probability $1-\alpha$. By the properties of the jointly controlled lottery, there cannot be profitable undetectable deviations at the stages where the jointly controlled lottery is played. For detectable deviations of the uninformed player 2(respectively 3) at the lottery stages, the informed player plays the optimal strategy of the zero-sum game $G_A(p_A)$ (respectively $G_B(p_B)$) to punish. For detectable deviations of the informed player at the lottery stages, the uniformed players play the approachability strategy of Theorem 6.5 to punish. The strategy profile where a jointly controlled lottery is played at initial stages – with deviations punished as described – and, after that, the corresponding strategy profile paying $\gamma_1$ or $\gamma_2$

drawn from the lottery, is a uniform equilibrium of the game. Indeed, we already showed that there are no profitable deviations during lottery stages. After lottery stages, players play a uniform equilibrium so there is no profitable deviation for any player, also. The ex-ante payoff of this equilibrium is $\alpha\gamma_1 + (1-\alpha)\gamma_2$.

**Proof of Proposition 3.9.** Let $(\sigma, \tau_A, \tau_B)$ be the equilibrium profile induced by the safe and independent joint-plan and let $(\beta_{A,s})_{s\in\mathbb{N}^*}$ and $(\beta_{B,s})_{s\in\mathbb{N}^*}$ be the expected payoffs martingales satisfying Lemma 5.3. Let also $(p_s)_{s\in\mathbb{N}^*}$ denote the process of posteriors associated to $(\sigma, \tau_A, \tau_B)$ (See section 5.2). First, note that by Definition 3.4, the martingale of posteriors has a.s. finite range. Hence there exists $s_0 \in \mathbb{N}^*$ such that $p_s$ is a.s. constant $\forall s \geq s_0$. Let $s \geq s_0$. Because the joint-plan is "safe" for players 2 and 3, it implies that $\beta_{A,s}(h_s) \leq v_A(p_s(h_s)_A)$ and $\beta_{B,s}(h_s) \leq v_B(p_s(h_s)_B), \forall h_s \in H_s$ that occurs with positive probability. Therefore $\beta_{A,s}(h_s) + \beta_{B,s}(h_s) \leq \mathrm{Cav}(\bar{h})(p_s(h_s))$ a.s., by definition of the concavification operator. Now, taking expectation on both sides over $h_s$ and using Jensen's Inequality, we have that $\beta_{A,1} + \beta_{B,1} \leq \mathrm{Cav}(\bar{h})(p)$. By Lemma 5.3, $\beta_{A,1} + \beta_{B,1}$ is the ex-ante payoff of the informed player. By Lemma 4.8, the result then follows.

**Lemma 6.11.** *Let $(\beta_{A,s})_{s\in\mathbb{N}^*}$ be the stochastic process of expected payoffs considered in Example 5.7. For each $s \in \mathbb{N}^*$, $\beta_{A,s} \leq V_A(p_{sA}) + Z_s$ a.s. where $(Z_s)_{s\in\mathbb{N}^*}$ is bounded, a.s. nonnegative and converges a.s. to 0.*

*Proof.* Fix $t, s \in \mathbb{N}^*$ with $t \geq s$. Conditioning over $\mathcal{H}_{t+1}$ and $\kappa_A$, we have $\mathbb{E}[A^{\kappa_A}_{i^t_A, j^t_A}|\mathcal{H}_s] = \mathbb{E}[\sum_{k_A\in K_A} p^{k_A}_{t+1A} A^{k_A}_{i^t_A, j^t_A}|\mathcal{H}_s] = \sum_{k_A\in K_A} p^{k_A}_{sA}\mathbb{E}[A^{k_A}_{i^t_A, j^t_A}|\mathcal{H}_s] + \sum_{k\in K}\mathbb{E}[(p^{k_A}_{t+1A} - p^{k_A}_{sA})A^{K_A}_{i^t_A, j^t_A}|\mathcal{H}_s] = \mathbb{E}[\bar{A}(p_{sA})_{i^t_A, j^t_A}|\mathcal{H}_s] + \sum_{k_A\in K_A}\mathbb{E}[(p^{k_A}_{t+1A} - p^{k_A}_{sA})A^{k_A}_{i^t_A, j^t_A}|\mathcal{H}_s] \leq V_A(p_{sA}) + \sum_{k_A\in K_A}\mathbb{E}[(p^{k_A}_{t+1A} - p^{k_A}_{sA})A^{k_A}_{i^t_A, j^t_A}|\mathcal{H}_s]$. Summing over $t \in \mathbb{N}^*$ from $s \leq t \leq T$, and dividing by $T$, we have

$$\mathbb{E}[\alpha_T|\mathcal{H}_s] \leq 2\frac{s}{T} + V_A(p_{sA}) + \frac{1}{T}\sum_{s\leq t\leq T}\sum_{k_A\in K_A}\mathbb{E}[|p^{k_A}_{t+1A} - p^{k_A}_{nA}||\mathcal{H}_s],$$

noting that the total payoffs up to $s$ are bounded by 1. Denote $Z_s := 2\frac{s}{T} + \frac{1}{T}\sum_{s\leq t\leq T}\mathbb{E}[\pi_s|\mathcal{H}_s]$, where $\pi_s := \sum_{k_A\in K_A}\sup_{t\geq s}|p^{k_A}_{t+1A} - p^{k_A}_{sA}|$. Taking Banach-limits (on $T$) on both sides, we have $\beta_{A,s} \leq V_A(p_{sA}) + \mathbb{E}[\pi_s|\mathcal{H}_s]$ a.s.. Since $p_s \to p^\infty$ a.s., as $s \to \infty$, it follows by Lemma 4.24 in Hart (1985), that $\mathbb{E}[\pi_s|\mathcal{H}_s] \to 0$ a.s.. ☐

Department of Economics, University of Rochester, NY 14627, USA.
*Email address*: pahl.lucas@gmail.com