O'NEILL'S THEOREM FOR GAMES

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ABSTRACT. We present the following analog of O'Neill's Theorem (Theorem 5.2 in [17]) for finite games. Let $C_1, \ldots, C_k$ be the components of Nash equilibria of a finite normal-form game G. For each i, let c_i be the index of C_i. For each $\varepsilon > 0$, there exist pairwise disjoint neighborhoods $V_1, ..., V_k$ of the components such that for any choice of finitely many distinct completely mixed strategy profiles $\{\sigma^{ij}\}_{ij}, \sigma^{ij} \in V_i$ for each i and numbers $r_{ij} \in \{-1, 1\}$ such that $\sum_j r_{ij} = c_i$, there exists a normal-form game $\bar{G}$ obtained from G by adding duplicate strategies and an ε-perturbation $\bar{G}^\varepsilon$ of $\bar{G}$ such that the set of equilibria of $\bar{G}^\varepsilon$ is $\{\bar{\sigma}^{ij}\}_{ij}$, where for each i, j: (1) $\bar{\sigma}^{ij}$ is equivalent to the profile σ^{ij}; (2) the index $\bar{\sigma}^{ij}$ equals r_{ij}.

1. INTRODUCTION

Multiplicity of equilibria is a pervasive phenomenon in games, and refinement theory aims to reduce it by strengthening the Nash criterion through the imposition of additional requirements. For most well-known refinement concepts, their power derives from asking for robustness of the equilibria with respect to some type of perturbation. Perfect [21], Proper [16] and Stable equilibria (either the Kohlberg-Mertens [10] or the Mertens [14] variants) are examples of concepts that require robustness of Nash equilibria with respect to perturbations of the players' strategies. Essentiality [23] and Hyperstability [10] are examples of concepts that require robustness to payoff perturbations. Strategy perturbations can be encoded in payoff perturbations, which implies that concepts that require robustness to payoff perturbations are usually more stringent than the strategic ones. In this paper we focus on payoff perturbations.

Every game induces an associated fixed-point problem, where the induced fixed-point map (usually called a Nash-map) or best-reply correspondence has as its fixed points the Nash equilibria of the game. Small perturbations of a game generate close-by perturbations of the fixed-point map, and therefore fixed-point theory can also inform us in the search for equilibria that are robust to payoff perturbations. The precise fixed-point theoretic tool that allows us to identify robustness of equilibria to perturbations of the fixed-point map is called the *fixed-point index*. The fixed-point index is an integer number associated to each connected component of fixed points of a map and provides a characterization of robustness: a component is robust to perturbations of its fixed-point map to nearby maps if and only if its index is nonzero.

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The lesson for game theory obtained from this characterization is somewhat ambiguous: though nonzero index components are robust to payoff perturbations (cf. [19]), the converse is not true (see [7]). Moreover, a fundamental result in fixed-point theory, O'Neill's Theorem (Theorem 5.2 in [17]) implies that any fixed-point in a component can be made the unique fixed point of a nearby map. Though this result permits the interpretation that no fixed point in a connected component of fixed points is particularly distinguishable from the point of view of robustness to perturbations of the map, it yields no immediate prescription for game theory in terms of equilibrium selection: maps that approximate a Nash map may not be Nash maps of payoff-perturbed games, and therefore the same lesson cannot be translated immediately to selection of equilibria through payoff robustness.

Govindan and Wilson [4] proved that if we allow for duplicate strategies, then the class of fixed point maps generated by payoff perturbations is rich enough to capture the notion of essentiality in fixed-point theory, i.e., of robustness to map-perturbations.[1] More precisely, they proved that a Nash component C of a finite game has index zero if and only if for any $\varepsilon > 0$, there exists an equivalent game and an ε-payoff perturbation of the equivalent game with no equilibria near the equivalent component to C. This result provides a clear picture of the relation between robustness to map perturbations and to payoff perturbations, but is still incomplete. What about equilibrium components of nonzero index? What is the structure of the equilibria generated by payoff perturbations around positive index components?

For general fixed-point problems, this question is answered by O'Neill's Theorem, which asserts that if $f : U \to X$ is a continuous function in a Euclidean neighborhood U of a fixed point component C of f, $r_1, \ldots, r_k$ are integers whose sum is the fixed-point index of C, and $x_1, \ldots, x_k$ are distinct points of C, then there is a map arbitrarily close to f whose fixed points are $x_1, \ldots, x_k$, with the fixed point index of each x_i being r_i. The result of O'Neill provides us with the precise topological invariant to understand which fixed-point components can be robust, namely, the index of a component of equilibria: for example, if a component of equilibria has an index of $+2$, one can select any finite number of points inside the component and assign to them integers such that, as long as the sum of the integers is $+2$, an arbitrarily nearby map exists with those points as the only fixed points around the component, and the indices allocated to them are precisely the integers we have chosen.

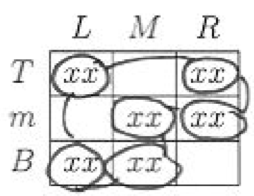
For game-theoretic problems, the immediate question is whether a similar addition of duplicate strategies and perturbation of payoffs of equivalent games would yield like in [4] an analog of O'Neill's Theorem for games. This is the main result we present in this paper. In essence, we prove that one can select in each equilibrium component a finite number of equilibria and associate to each one of them an integer equal to $+1$ or -1, such that the sum of the numbers allocated to the

[1]The addition of duplicate strategies is not only of technical importance to bridge the gap between the space of maps and that of payoff perturbations in the result of [4], but is important in the theory of refinements as a decision-theoretic property: Kohlberg and Mertens list the property of *Invariance* as a requirement a solution concept should satisfy, in the sense that a solution concept should not depend on the addition/elimination of duplicate strategies to the game, i.e., the solution should depend only on the reduced normal form.

selected points of a same component equals its index. For each $\varepsilon > 0$, one can then construct an equivalent game—obtained by adding duplicate strategies to the original game—and an ε-payoff perturbation of this equivalent game with a single Nash equilibrium close to each selected point, each of them having an index precisely equal to the pre-assigned number.

Let us illustrate this result in an example. Consider the following bi-matrix game (from Kohlberg and Mertens [10]). It has a unique Nash (connected) component of equilibria, homeomorphic to a circle.

	L	M	R
t	$(1,1)$	$(0,-1)$	$(-1,1)$
m	$(-1,0)$	$(0,0)$	$(-1,0)$
b	$(1,-1)$	$(0,-1)$	$(-2,-2)$



Say we would like to make $(\frac{1}{2}t + \frac{1}{2}b, L)$ the unique equilibrium (and so necessarily with index $+1$) of a perturbed equivalent game, we duplicate strategy L and perturb the payoffs as follows.

	L	L'	M	R
t	$(1+\varepsilon,1)$	$(1,1+\varepsilon)$	$(0+\varepsilon,-1)$	$(-1+\varepsilon,1)$
m	$(-1,0)$	$(-1,+\varepsilon)$	$(0,0)$	$(-1,0)$
b	$(1,-1+\varepsilon)$	$(1+\varepsilon,-1)$	$(0,-1)$	$(-2,-2)$

As m, M and R are strictly dominated in the above game, eliminating them leads to a two-by-two game where the unique Nash equilibrium is $(\frac{1}{2}t + \frac{1}{2}b; \frac{1}{2}L + \frac{1}{2}L')$ (necessarily with index $+1$) which is equivalent in the original game to $\sigma = (\frac{1}{2}t + \frac{1}{2}b, L)$. Consider now the three Nash equilibria (t, L) and (b, L) and $\sigma = (\frac{1}{2}t + \frac{1}{2}b, L)$ and associate to the first two the integer $+1$ and to the last the integer -1. We will show how to add to the original game the same duplicate strategy L' as above and perturb the payoffs so as to generate exactly three equilibria in a perturbed game. Consider the following perturbation:

	L	L'	M	R
t	$(1+\varepsilon,1+\varepsilon)$	$(1,1)$	$(0+\varepsilon,-1)$	$(-1+\varepsilon,1)$
m	$(-1,0)$	$(-1,+\varepsilon)$	$(0\,0)$	$(-1,0)$
b	$(1,-1)$	$(1+\varepsilon,-1+\varepsilon)$	$(0,-1)$	$(-2,-2)$

Strategies m, M and R are strictly dominated. Eliminating them leads to a battle of the sexes where only three Nash equilibria are present: the two strict equilibria (t, L) and (b, L') (hence with indices $+1$ each) and a completely mixed equilibrium $(\frac{1}{2}t + \frac{1}{2}b; \frac{1}{2}L + \frac{1}{2}L')$ with index -1. But since L' was a duplicate of L, we obtain as Nash equilibria (t, L) and (b, L) and $(\frac{1}{2}t + \frac{1}{2}b, L)$, and their indices correspond to the integers we fixed previously. Hence, perturbations to nearby equivalent games can single out σ as an isolated equilibrium with different indices: in the first perturbation we obtained it with index $+1$ and in the second with index -1.

Our proof of the extension of O'Neill's Theorem to games is inspired by the one in Govindan Wilson [4]. Though the steps of the proof are similar, the details are very different and pose different technical challenges. It requires more preliminary work in the realm of fixed point theory in order to produce a suitable approximation of the best-reply correspondence (which is achieved with the help of a fundamental result in obstruction theory, namely, the Hopf Extension Theorem). Then we construct a game that has essentially that approximation as its best-reply correspondence (which will require tools from the theory of triangulations).

The paper is organized as follows. Section 2 sets up the problem and states the main result. Section 3 gives a detailed outline of the proof and includes a table of notation that is used in the proof. Section 4 contains the proof of the theorem, which is presented in a sequence of steps.

2. Preliminary Definitions and Statement of the Theorem

We start by introducing some notational conventions. A finite normal-form game is a tuple $G = (\mathcal{N}, (S_n)_{n\in\mathcal{N}}, (G_n)_{n\in\mathcal{N}})$, where $\mathcal{N} = \{1, ..., N\}$ is the set of players, S_n is a finite set of pure strategies of player n, $S \equiv \times_{n\in\mathcal{N}} S_n$ and $G_n : S \to \mathbb{R}$ the payoff function of player n. We denote the set of mixed strategies of player n by $\Sigma_n \equiv \Delta(S_n)$ and the set of mixed-strategy profiles is denoted $\Sigma \equiv \prod_n \Sigma_n$. For each n, we will continue to use G_n to denote tihe extension of n's payoff function to Σ. The best-reply correspondence of G is denoted BR.

For $V_n \subseteq \Sigma_n$, ∂V_n denotes the boundary of V_n in the affine space generated by Σ_n; $\partial_{\Sigma_n} V_n$ refers to the boundary of V_n in Σ_n. Similarly, ∂V and $\partial_\Sigma V$ denote, resp., the boundaries of $V \subseteq \Sigma$ in the affine space generated by Σ and in Σ. Finally, $\|\cdot\|$ denotes the ℓ_∞-norm on Euclidean spaces.

A *polyhedral subdivision* $\mathcal{P}$ of a polytope $X \subset \mathbb{R}^m$ is a collection of polytopes in $\mathbb{R}^m$ that satisfies two conditions: first, the union of the polytopes in $\mathcal{P}$ is equal to X; second, the intersection of any two polytopes in $\mathcal{P}$ is a face of each (possibly empty). Given a polyhedral subdivision $\mathcal{P}$ of X, a typical member of $\mathcal{P}$ is called a *cell*. The cells of dimension 0 are called vertices. When the polyhedral subdivision is made of simplices, it is called a *triangulation*. The *closed star of a vertex v of a triangulation $\mathcal{T}_n$* is the union of simplices that have v as a vertex; and the *simplicial neighborhood* of the closed star of v is the union of all simplices that intersect the closed star.

The *face complex* of a polytope $X \subset \mathbb{R}^m$ is the polyhedral subdivision $\mathcal{P}_n$ of X which is given by the collection of its faces. If $\mathcal{P}_n$ is any polyhedral subdivision of X, then a *subcomplex* of $\mathcal{P}_n$ is a subset of $\mathcal{P}_n$ that is itself a polyhedral subdivision.

For technical reasons, our proof will require considering the notion of a game defined over slightly more general strategy sets than simplices for each player (cf. Pahl [18]).

Definition 2.1. A *polytope-form game* is a tuple $G = (\mathcal{N}, (P_n)_{n\in\mathcal{N}}, (V_n)_{n\in\mathcal{N}})$, where $\mathcal{N} = \{1, ..., N\}$ is the set of players, P_n is a polytope in $\mathbb{R}^{m_n}$ (the set of strategies of player n) and $G_n : \prod_{n\in\mathcal{N}} P_n \to \mathbb{R}$ is the payoff function of player n, which is affine in each coordinate $p_j \in P_j, j = 1, ..., N$.[2]

Obviously, every finite game in mixed strategies can be viewed as a game in polytope-form since strategy sets are simplices hence polytopes and payoff functions are multi-linear. Going the other way, given a polytope-form game, we can define a normal-form game where the set of pure strategies of each player is the set of vertices of P_n; these two are "equivalent" in the sense to be defined now.

Definition 2.2. For $G = (\mathcal{N}, (P_n)_{n\in\mathcal{N}}, (G_n)_{n\in\mathcal{N}})$ and $G' = (\mathcal{N}, (P'_n)_{n\in\mathcal{N}}, (G'_n)_{n\in\mathcal{N}})$ polytope-form games, G and G' are *equivalent* if there exist a polytope-form game $\tilde{G} = (\mathcal{N}, (\tilde{P}_n)_{n\in\mathcal{N}}, (\tilde{G}_n)_{n\in\mathcal{N}})$, affine and surjective maps $\phi_n : P_n \to \tilde{P}_n$ and $\phi'_n : P'_n \to \tilde{P}_n$, such that $G_n(p) = \tilde{G}_n(\phi(p))$ and $G'_n(p') = \tilde{G}_n(\phi'(p'))$, with $\phi \equiv \times_n \phi_n, \phi' \equiv \times_n \phi'_n$.

Let G and $\tilde{G}$ be as in Definition 2.2. We say $p_n \in P_n$ *projects* to $\tilde{p}_n \in \tilde{P}_n$, if $\phi_n(p_n) = \tilde{p}_n$; similarly, we say $p \in P$ projects to $\tilde{p} \in \tilde{P}$, if $\phi(p) = \tilde{p}$. We say that a set $A \subseteq P$ projects to $\tilde{A} \subseteq \tilde{P}$ if $\phi(A) = \tilde{A}$. Given a normal-form game G, a *duplicate strategy* s_n *of player* n is a pure strategy that projects to a (possibly mixed) strategy of player n that is distinct from s_n. A normal-form game $\bar{G}$ obtained by adding duplicate strategies from G is a finite game where each pure strategy is either a pure strategy of G or is a duplicate of a strategy of G.

2.1. **Fixed Point Index.** We provide a brief review of the concept of fixed point index here. For a comprehensive and axiomatic treatment of the subject we refer the reader to [13]. Let X be a convex set and A a compact subset of X. Say that a correspondence $\varphi : A \twoheadrightarrow X$ is well-behaved if it is nonempty, compact, convex-valued and upper semicontinuous. Suppose φ is a well-behaved correspondence without fixed points on the boundary of A (in X). The index of φ (over A) is an integer that serves as an algebraic count of the number of fixed points of f in A. In the case where A is the closure of an open subset of an Euclidean space, and φ is a function, the index of φ can be computed as the degree of the displacement d_φ of φ, which is defined as $d_\varphi \equiv \mathrm{Id} - \varphi$.

We invoke three important properties of the fixed point index. The first property is continuity: there exists $\varepsilon > 0$ such that for all well-behaved correspondences $\varphi' : A \twoheadrightarrow X$ whose graph is in the ε-neighborhood of that of φ, the index of φ' is the same as that of φ. Since a well-behaved correspondence φ can be approximated by functions whose graphs are contained in arbitrarily small neighborhoods of the graph of φ, the index of φ can be computed by approximating it by a function.

[2]The original term to denote this class of games is *strategic-form games* and was coined in [15]. In order to avoid confusion with the current terminology of "strategic-form games", we opted for the "polytope-form game" designation.

The second property, which is related to continuity, is homotopy. Given two well-behaved correspondences $\varphi, \varphi' : A \twoheadrightarrow X$, if $\lambda\varphi + (1-\lambda)\varphi$ has no fixed points on the boundary of A for any $\lambda \in [0,1]$, then the index of φ and φ' are the same.

The third property we need is the multiplicative property. If $\varphi_1 : A_1 \twoheadrightarrow X_1$ and $\varphi_2 : A_2 \twoheadrightarrow X_2$ are two-well behaved correspondences without fixed points on the boundaries of A_1 and A_2, the index of $\varphi_1 \times \varphi_2$ is the product of the indices of φ_1 and φ_2.

Suppose $\varphi : X \twoheadrightarrow X$ is a well-behaved correspondence and X is compact and convex. Let C be a component of fixed points of φ. Let A be a closed neighbrohood of C that contains no other fixed points. The index of C is defined to be the index of φ over A. The index of C is independent of the choice of the neighborhood, so long as the neighborhood does not contain any other fixed points.

Given a game G in polytope-form, let $\varphi : P \twoheadrightarrow P$ be the best-reply correspondence, i.e., $\varphi(p)$ is the set of $q \in P$ such that for each n, q_n is a best reply to p. φ is a well-behaved correspondence. The game G has finitely many components of Nash equilibria, obtained as the set of fixed points of φ. For each component C, we can then assign an index. The index of Nash equilibria can also be computed using any Nash map, i.e., a function f that is jointly continuous in payoffs and strategies and whose fixed points for any game are the Nash equilibria (cf. [3]).

2.2. **Statement of the Theorem.** Fix a finite normal-form game G. Let $C_1, \ldots, C_k$ be the components of its equilibria. Let the index of each C_i be c_i.

Theorem 2.3. *For each $\varepsilon > 0$, there exist pairwise disjoint sets $V_1, \ldots, V_k \subseteq \Sigma$, V_i a neighborhood of C_i for each i, such that for any choice of finitely many distinct points $\{\sigma^{ij}\}_{ij}, \sigma^{ij} \in V_i \setminus \partial V_i$ and numbers $r_{ij} \in \{-1, 1\}$ such that $\sum_j r_{ij} = c_i$, there exists a normal-form game $\hat{G}$ that is equivalent to G and a ε-perturbation $\hat{G}^\varepsilon$ of $\hat{G}$ such that:*

(1) The set of equilibria of $\hat{G}^\varepsilon$ is finite and equals $\{\hat{\sigma}^{ij}\}_{ij}$, where $\hat{\sigma}^{ij}$ projects to σ^{ij}, for each i, j.
(2) The index of $\hat{\sigma}^{ij}$ equals r_{ij}.

The following corollary follows immediately from Theorem 2.3 and distinguishes components with a positive index from the others. Below, we say that a game is *generic* if it possesses finitely many equilibria, each with index $+1$ or -1. It is known that the set of payoffs of a game for which this property is violated has a strictly lower dimension than the whole payoff space.

Corollary 2.4. *A set $C \subseteq \Sigma$ of strategy profiles is a component of equilibria of G with a positive index iff it is a minimal set (under set inclusion) with the following property: for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each equivalent game $\bar{G}$ and each generic $\bar{G}^\delta$ that is an δ-perturbation of $\bar{G}$, there is a $+1$ equilibrium whose projection to Σ is within ε of C.*

Corollary 2.4 shows that $+1$ equilibria always arise in generic perturbations of payoffs of a finite game, locally around a positive index component. In contrast, when negative index components are considered, there are always generic payoff perturbations for which all the equilibria arising locally have -1 index. The relevance of this is that $+1$ equilibria possess a number of interesting properties their -1 counterparts do not: [8] shows that the $+1$ equilibria of a generic game are the only ones that are dynamically stable for at least one Nash dynamics; in [6], we show that the $+1$ equilibria are the only *sustainable* ones. [12] argues for the selection of $+1$ index equilibria, on both theoretical as well as experimental grounds, and [9] uses the $+1$ index selection criterion to refine equilibria in signaling games.

3. A Guide to the Proof of Theorem 2.3

This section provides a rough sketch of the proof of Theorem 2.3, which is presented formally in the next section. The main objective here is to highlight the key ideas involved in modifying O'Neill's theorem to construct a map that is game-theoretically meaningful, i.e., whose fixed points are Nash equilibria of a game.

In order to leverage index theory, we work with the best-reply correspondence, rather than a Nash map. The main reason is that for any approximation of the best-reply correspondence by a continuous function f, we have immediate information on the degree of suboptimality of $f(\sigma)$ against σ. More precisely, fix $\varepsilon > 0$ as in the statement of Theorem 2.3. Let BR_ε be the ε-best reply correspondence, i.e., $\mathrm{BR}_\varepsilon(\sigma)$ is the set of all profiles τ such that for n, τ_n yields a payoff against σ that is strictly within ε of the best payoff achievable against σ. The graph of the ε-best-reply correspondence, denoted $\mathrm{Graph}(\mathrm{BR}_\varepsilon)$, is a neighborhood of the graph of BR. For any continuous function f whose graph is contained in $\mathrm{Graph}(\mathrm{BR}_\varepsilon)$, $f(\sigma)$ is an ε-best-reply to σ.

The sets V_i identified in the statement of the theorem are neighborhoods of the components that are pairwise disjoint and such that for any σ that belongs to some V_i, $(\sigma, \sigma) \in \mathrm{Graph}(\mathrm{BR}_\varepsilon)$. Now we fix points σ^{ij} as in Theorem 2.3. Our first task, carried out in Step 2, is to obtain a version of O'Neill's result by constructing an approximation of BR that has the σ^{ij}'s as its fixed points and with the assigned indices. This proof calls for a minor modification of the proof in O'Neill's paper that is necessitated by two key points of departure of our set up from his. (1) We are dealing with a correspondence and not a function; (2) Our components of fixed points need not lie in the interior of the strategy space.[3] To handle these problems, we approximate BR by a function $\hat{f}$ whose graph is in $\mathrm{Graph}(\mathrm{BR}_\varepsilon)$ and whose fixed points are in $\cup_i V_i \setminus (\partial\Sigma \cup \partial V_i)$. Now, we modify the function inside the V_i's to obtain a function f whose graph is still in $\mathrm{Graph}(\mathrm{BR}_\varepsilon)$ but whose fixed points are the σ^{ij}'s with index r^{ij}.

Since we want to use f to mimic a best-reply correspondence, the analysis is made easier if the function f is locally affine around the σ^{ij}'s. This yields us the structure of a polymatrix game (one

[3]O'Neill takes a component of fixed points isolated in an Euclidean neighborhood—i.e., a set homeomorphic to a neighborhood in an Euclidean space. If our component of fixed points lies in the interior of the strategy space, then we could take a neighborhood of it in the interior of the strategy space, which is obviously an Euclidean neighborhood. However, if the component intersects the boundary, this is not immediately possible.

where the payoffs of a player are linear in the strategies of his opponents) at least locally around the equilibria of the game $\hat{G}^{\varepsilon}$, specified in the theorem. The job of Step 1 is to ensure that it is possible to make f locally affine.

We would like to view f as a proxy for a best-reply correspondence of a perturbation of an equivalent game, as that would indicate how to perturb payoffs. There are two hurdles to doing so, which we deal with sequentially in our proof. The first problem is that the function f maps into $\Sigma \setminus \partial\Sigma$ and therefore if $f(\sigma)$ is a "best-reply" to σ, then every pure strategy is one as the set of best replies is a face of the strategy set. The way around this problem is to say that $f(\sigma)$ is equivalent to a strategy in an equivalent game $\tilde{G}$, thus avoiding making everything a best-reply. Constructing a finite equivalent game then requires us to introduce finitely many points in each Σ_n as duplicate pure strategies and making $f(\sigma)$ mixtures over these points. Geometrically speaking, this leads us to a triangulation $\mathcal{T}_n$ of Σ_n as carried out in Step 3. For each σ, player n's "best replies" to σ are among the vertices of the simplex that contains $f_n(\sigma)$ in its interior.

The second problem is that, unlike a best-reply correspondence, the coordinate function f_n of player n could depend on his strategy, σ_n.[4] To deal with this problem, we consider an equivalent polytope-form game in which each player n chooses a strategy in Σ_n and one in Σ_{n+1} (where $n+1 = 1$ if $n = N$), and the strategy space of player n is $\Sigma_n \times \Sigma_{n+1}$. The construction is such that the choice by n of a strategy in Σ_{n+1} is payoff-irrelevant for player n. Now the function f can be "lifted" to this strategy space by making f_n depend on σ_{-n} and player $n-1$'s choice in Σ_n (where $n-1 = N$ if $n = 1$). Combining this idea, along with the fact that the triangulation $\mathcal{T}_n$ introduces new pure strategies for n, we introduce a G-equivalent polytope-form game $\tilde{G}$ in Step 4. Letting $\tilde{S}_n$ be the set of vertices of the triangulation $\mathcal{T}_n$ from Step 3, the strategy set of n in $\tilde{G}$ is the product $\Delta(\tilde{S}_n) \times \Delta(\tilde{S}_{n+1})$ (a polytope). There is a map $\tilde{f}$ defined on this strategy space that enjoys properties analogous to f, with the crucial difference being that n's component maps are independent of n's strategy. In this step, we also identify distinguished points $\tilde{\sigma}^{ij}$ that are equivalent to σ^{ij}.

Step 5 shows the sense in which the function $\tilde{f}$ is indeed a proxy for a perturbed best-reply correspondence. For each $\tilde{\sigma}$, we define a perturbation of the polytope-form game $\tilde{G}$ (constructed in step 4) such that for each n the best replies to $\tilde{\sigma}$ are among the vertices of the simplex that contains $\tilde{f}_n(\tilde{\sigma})$ in its interior. The perturbation comes in the form of a bonus $\tilde{g}_n^0$ for the first coordinate of n's strategy and $\tilde{g}_n^1$ for the second coordinate, i.e., in the perturbed game there are payoff components $\tilde{g}_n^0$ and $\tilde{g}_n^1$ for each n that depend only on his choices in $\tilde{S}_n$ and $\tilde{S}_{n+1}$.

Step 6 shows that the only profiles that are equilibria of the perturbed games constructed in step 5 are the distinguished points $\tilde{\sigma}^{ij}$'s of step 4 and that their fixed-point indices in the perturbed games are equal to r^{ij} (as we want).

[4]One could say that the problem is more acute around the σ^{ij}'s where f_n is exclusively a function of σ_n!

Steps 5 and 6 provide a continuous family of payoff perturbations, indexed by strategies. The goal of the next two steps is to define one perturbed game. Just as in Step 3 where we introduced a triangulation $\mathcal{T}_n$ to add their vertices as duplicate strategies, we now introduce in Step 7 a triangulation $\tilde{\mathcal{T}}_n$ of the strategy set of player n in $\tilde{G}$ to approximate the continuous family by using the values of the perturbations at the vertices of the triangulation $\tilde{\mathcal{T}}_n$. The step also shows the existence of certain piecewise-linear, convex functions that are used in Step 8 to construct bonus functions. Step 8 introduces an equivalent game $\hat{G}$ where the strategy set of player n consists of a pair of vertices, one for $\tilde{T}_n$ and the other for $\tilde{\mathcal{T}}_{n+1}$. The triangulation $\tilde{\mathcal{T}}_n$ is used to discretize the bonus function $\tilde{g}$. In order to make sure that each player n only randomizes over strategies that are vertices of simplices, the piecewise-linear function described above is added as perturbations $\hat{g}^0$ and $\hat{g}^1$ as well. Finally, an extra function $\hat{G}^*$ is added to eliminate the multiplicity of equilibria, as otherwise, there is a component of strategies that are all equivalent to $\tilde{\sigma}^{ij}$.

Step 9 concludes the proof by exploiting the fact that this approximation by a finite game has the relevant properties of the continuous family of perturbations.

Since the proof involves a lot of notation, some of which are step-specific and others not, we end this section with a table of the notations that are used across multiple steps.

Table 1. Table of Notations

Notation	**Reference and/or Definition**
Y_n^{ij}	Simplex contained in Σ_n, with σ_n^{ij} as barycenter (see Step 1)
X_n^{ij}	Simplex contained in Y_n^{ij}, with σ_n^{ij} as barycenter (see Step 1)
f_n^{ij}	$f_n^{ij} : X_n^{ij} \to Y_n^{ij}$; see Lemma 4.1
BR	Best-reply correspondence of game G
BR_ε	ε-Best-reply correspondence of game G (see Step 1)
f	$f : \Sigma \to \Sigma$ (See Lemma 4.2, Step 2)
$\mathcal{T}_n$	Triangulation of Σ_n (see Lemma 4.3, Step 3)
T_n^{ij}	Simplex of $\mathcal{T}_n$ contained in X_n^{ij}, containing σ_n^{ij} as its barycenter (see Lemma 4.3, Step 3)
$\tilde{S}_{n,0}$	Vertices of the triangulation $\mathcal{T}_n$ (see Step 4)
$\tilde{S}_{n,1}$	Vertices of the triangulation $\mathcal{T}_{n+1}$ (see Step 4)
$\tilde{\Sigma}_{n,k}$	$\tilde{\Sigma}_{n,k} \equiv \Delta(\tilde{S}_{n,k})$ (See Step 4)
$\tilde{G}$	Polytope-form game where the mixed strategy set of each player n is $\tilde{\Sigma}_n \equiv \tilde{\Sigma}_{n,0} \times \tilde{\Sigma}_{n,1}$ (see Step 4)
$\tilde{B}R_\varepsilon$	ε-Best-reply correspondence of game $\tilde{G}$ (see Step 4)
$\tilde{\beta}_{n,k}$	$\tilde{\beta}_{n,k} : \Sigma_n \to \tilde{\Sigma}_{n,k}$ is the barycentric map: it associates to a point σ_n its barycentric coordinates in the triangulation $\mathcal{T}_n$, viewed as a face of $\tilde{\Sigma}_{n,k}$ (see Step 4)
$\phi_{n,k}$	$\phi_{n,k} : \tilde{\Sigma}_{n,k} \to \Sigma_{n+k}$ projects the point $\tilde{\sigma}_{n,k}$ to its location in Σ_{n+k} (see Step 4)
$\tilde{\Sigma}^*_{n,k}$	$\tilde{\Sigma}^*_{n,k} \equiv \tilde{\beta}_{n,k}(\Sigma_{n+k})$ (see step 4)
$\mathcal{S}^*_{n,k}$	Subcomplex of the face complex $\Delta(\tilde{S}_{n,k})$ whose space is $\tilde{\Sigma}^*_{n,k}$ (see Step 4)
$\tilde{T}^{ij}_{n,k}$	$\tilde{T}^{ij}_{n,k} \equiv \tilde{\beta}_{n,k}((f^{ij}_{n+k})^{-1}(T^{ij}_{n+k}))$ (see Step 4)

Notation	**Reference and/or Definition**
$\tilde{f}_{n,0}$	$$\tilde{f}_{n,0} : \tilde{\Sigma}_{-n} \to \tilde{\Sigma}_{n,0},$$ $$\tilde{f}_{n,0}(\tilde{\sigma}_{-n}) \equiv \tilde{\beta}_{n,0}(f_n(\phi_{-n,0}(\tilde{\sigma}_{-n,0}), \phi_{n-1,1}(\tilde{\sigma}_{n-1})))$$ (see Step 4 and Lemma 4.4)
$\tilde{f}_{n,1}$	$\tilde{f}_{n,1}(\tilde{\sigma}_{n+1}) \equiv \tilde{\beta}_{n,1}(\phi_{n+1,0}(\tilde{\sigma}_{n+1}))$ (see Step 4 and Lemma 4.4)
$\tilde{g}^0(\cdot)$	$\tilde{g}^0 : \tilde{\Sigma} \to \prod_n \mathbb{R}^{\tilde{S}_{n,0}}$ (see Step 5)
$\tilde{g}^1(\cdot)$	$\tilde{g}^1 : \tilde{\Sigma} \to \prod_n \mathbb{R}^{\tilde{S}_{n,1}}$ (see Step 5)
$r_n(\cdot)$	$r_n : \Sigma_{-n} \to \mathbb{R}$ returns the best payoff player n obtains against σ_{-n} (see Step 5)
$\tilde{\mathcal{T}}_n$	Triangulation of $\Delta(\tilde{S}_{n,0}) = \Delta(\tilde{S}_{n-1,1})$ (see Step 7)
$\hat{S}_{n,0}$	$\hat{S}_{n,0}$ be the set of vertices of $\tilde{\mathcal{T}}_n$, $\hat{S}_{n,0} = \hat{S}_{n-1,1}$ (see Step 8)
$\hat{T}^{ij}_n$	Carrier of $\tilde{\sigma}^{ij}_n$ in $\tilde{\mathcal{T}}_n$ (see Step 7 and Lemma 4.8)
$\hat{\phi}_{n,k}$	$\hat{\phi}_{n,k} : \hat{\Sigma}_{n,k} \to \tilde{\Sigma}_{n,k}$ projects strategies from $\hat{\Sigma}_{n,k}$ to $\tilde{\Sigma}_{n,k}$ (see Step 8)
$\hat{\mu}_{n,k}$	$\hat{\mu}_{n,k} : \hat{\Sigma}_n \to \hat{\Sigma}_{n,k}$ computes the marginal of $\hat{\sigma}_n$ over $\Sigma_{n,k}$ (see Step 8)
$\hat{G}$	Normal-form game with pure strategy set of player n equals to $\hat{S}_n \equiv \hat{S}_{n,0} \times \hat{S}_{n,1}$ and payoff $\hat{G}_n(\hat{\sigma}) = \hat{G}_n(\hat{\phi} \circ \hat{\mu}(\hat{\sigma}))$ (see Step 8)

4. Proof of Theorem 2.3

As in the statement of the theorem, we have the following notation, used uniformly throughout the section. For each component C_i, $i = 1, \ldots, k$, we have a finite (possibly empty) set J_i, and a finite set of integers $r^{ij} \in \{+1, -1\}$, $j \in J_i$, such that $\sum_{j \in J_i} r^{ij} = c_i$, with the sum being zero if J_i is an empty set.

Step 1. Fix a finite number of completely-mixed strategy profiles σ^{ij}, $i = 1, \ldots, k$ and $j \in J_i$ such that for each n the σ^{ij}_n's are distinct mixed strategies. Locally around each σ^{ij}, we construct an affine map that has σ^{ij} as its unique fixed point and assigns it the index r^{ij}. In Step 2, these maps are then extended to the whole of Σ (not affinely though) without introducing additional fixed points.

For each n, and i, j, let X^{ij}_n and Y^{ij}_n be two full-dimensional simplices in Σ_n with σ^{ij}_n as their barycenters. Let $X^{ij} = \prod_n X^{ij}_n$ and $Y^{ij} = \prod_n Y^{ij}_n$. We will assume that the simplices Y^{ij}_n are small enough that the Y^{ij}'s are pairwise disjoint.

Lemma 4.1. *For each i, j and n, there is an affine homeomorphism $f^{ij}_n : X^{ij}_n \to Y^{ij}_n$ such that:*

(1) σ_n^{ij} *is the unique fixed point of* f_n^{ij}*;*

(2) letting $f^{ij} = \prod_n f_n^{ij} : X^{ij} \to Y^{ij}$*, the index of* σ^{ij} *under* f^{ij} *is* r^{ij}*.*

Proof. The result for the case $N = 1$ was proved in [5]. Use their result for each n, assigning index r^{ij} to player 1's map f_1^{ij} and $+1$ for the maps of all other players. The index of σ^{ij} under f^{ij} is r^{ij} by the multiplication property of the index. □

Step 2. This step gives us a version of O'Neill's theorem with three main differences. (1) It applies to the BR correspondence. (2) The points chosen are in the interior of Σ and close to, but not necessarily in, C_i. (3) We insist that the map be locally affine around the fixed points.

Let Graph(BR) be the graph of the best-reply correspondence $\mathrm{BR} : \Sigma \twoheadrightarrow \Sigma$ of the game G. For each $\varepsilon > 0$, let BR_ε be the ε-best-reply correspondence, i.e., for each σ, $\mathrm{BR}_\varepsilon(\sigma)$ is the set of strategy profiles τ such that for each n, $G_n(\sigma_{-n}, \tau_n) > \max_{s_n} G_n(\sigma_{-n}, s_n) - \varepsilon$. The graph of BR_ε, denoted Graph(BR_ε), is a neighborhood of Graph(BR). (In fact, the sets Graph(BR_ε) form a basis of neighborhoods of Graph(BR).) From now on, we fix $\varepsilon > 0$ as required by the main theorem.

Choose $\delta > 0$ such that the δ-neighborhoods V_i of C_i satisfy the following properties: $V_i \cap V_j = \emptyset$ for all $i \neq j$; and for all i and $\sigma \in V_i$, (σ, σ) belongs to Graph(BR_ε). These V_i's are the ones specified in the statement of Theorem 2.3. These, too, are fixed from now on. Pick points σ^{ij}, $i = 1, \ldots, k$, $j \in J_i$ as required by the same theorem and fix them from now on as well. If necessary by considering an equivalent game $\bar{G}$ obtained by adding duplicate strategies, we can assume that the σ_n^{ij}'s are all distinct for each n. Indeed, consider an equivalent game $\bar{G}$ where we duplicate every pure strategy of every player. Let $\bar{\Sigma}$ be the strategy space in $\bar{G}$ and $\phi : \bar{\Sigma} \to \Sigma$ be the map that sends strategies in $\bar{G}$ to equivalent profiles in G. Let $\bar{C}_i = \phi^{-1}(C_i)$ be the equilibrium component of $\bar{G}$ for each i, and let $\bar{V}_i \equiv \phi^{-1}(V_i)$ be the corresponding neighborhood of $\bar{C}_i$. For each σ^{ij}, we can pick a point $\bar{\sigma}^{ij} \in \phi^{-1}(\sigma^{ij})$ such that for each n, the $\bar{\sigma}_n^{ij}$'s are distinct mixed strategies. Because $\bar{G}$ is equivalent to G, each $\bar{C}_i$ has the same index as C_i (cf. Theorem 5 in [4]). Therefore, in what follows, we can replace G with $\bar{G}$. To simplify notation, we will refer to the game still as G and just assume that the chosen σ^{ij}'s involve distinct mixed strategies for each player.

Lemma 4.2. *There exists a continuous function* $f : \Sigma \to \Sigma$ *with the following properties:*

(1) the graph of f *is contained in Graph(*BR_ε*);*

(2) the fixed points of f *are the points* σ^{ij} *and the index of each* σ^{ij} *is* r^{ij}*;*

(3) for each n *and* i, j*, there exist a polyhedral subdivision* $\mathcal{X}_n$ *of* Σ_n *and full-dimensional simplices* X_n^{ij} *and* Y_n^{ij} *such that:*

  (a) σ_n^{ij} *is the barycenter of both* X_n^{ij} *and* Y_n^{ij}*;*

  (b) $X_n^{ij} \subset \mathrm{int}(Y_n^{ij}) \subset \Sigma_n \setminus \partial\Sigma_n$*;*

  (c) $\prod_n Y_n^{ij} \times \prod_n Y_n^{ij} \subset$ *Graph(*BR_ε*);*

(d) *the restriction* f^{ij} *of* f *to* $\prod_n X_n^{ij}$ *decomposes into a product map* $\prod_n f_n^{ij}$ *where* f_n^{ij} *is an affine homeomorphism of* X_n^{ij} *onto* Y_n^{ij}*;*

(e) X_n^{ij} *is a subcomplex of* $\mathcal{X}_n$ *and the carrier of* σ_n^{ij} *in* $\mathcal{X}_n$ *has dimension* $\dim(\Sigma_n)$*.*

Proof. By Corollary 1 and Axiom 2 in [11], there exists a function $\hat{f} : \Sigma \to \Sigma$ such that

(1) the graph of $\hat{f}$ is contained in $\mathrm{Graph}(\mathrm{BR}_\varepsilon)$;
(2) all its fixed points are contained in $\cup_i (V_i \setminus \partial_\Sigma V_i)$;
(3) the index of $\hat{f}$ over V_i is c_i for each i.

If necessary by replacing $\hat{f}$ with a suitable convex combination $(1-\alpha)\hat{f} + \alpha\sigma^\circ$, where σ° is a completely mixed profile in Σ and $\alpha > 0$ is sufficiently small, we can assume furthermore that the fixed points of $\hat{f}$ are contained in $\Sigma \setminus \partial\Sigma$.

Let $C_i(\hat{f})$ be the set of fixed points of $\hat{f}$ in V_i. As V_i is the δ-neighborhood of the component C_i, it is semialgebraic (cf. Proposition 2.2.8 in [1]) and $V_i \setminus (\partial_{\Sigma_i} V_i \cup \partial\Sigma)$ is connected. Therefore for all $\eta > 0$ sufficiently small, the set of points in V_i whose distance from $\partial_{\Sigma_i} V_i \cup \partial\Sigma$ is strictly greater than η is a connected set. As the points σ^{ij} and the set $C_i(\hat{f})$ lie in $V_i \setminus (\partial_{\Sigma_i} V_i \cup \partial\Sigma)$, we can choose $\eta > 0$ small enough such that for each i, letting U_i be the set of $\sigma \in V_i$ such that $d(\sigma, \partial_{\Sigma_i} V_i \cup \partial\Sigma) > 2\eta$, we have that $U_i \setminus \partial U_i$ is connected and contains $C_i(\hat{f})$ as well as the points σ^{ij}. Furthermore, by choosing an even smaller η, if necessary, we can assume that $(\sigma, \tau) \in \mathrm{Graph}(\mathrm{BR}_\varepsilon)$ for $\sigma \in V_i$ and τ within 2η of σ.

Use Urysohn's lemma to construct a function $\gamma : \Sigma \to [\eta, 1]$ that equals 1 outside the interiors of V_i's and η on U_i. Replace the function $\hat{f}$ with the function $\hat{f}^1 \equiv \gamma\hat{f} + (1-\gamma)\mathrm{Id}$. The graph of $\hat{f}^1$ is also contained in $\mathrm{Graph}(\mathrm{BR}_\varepsilon)$ and its fixed points are still the points in the $C_i(\hat{f})$'s. Also, $\|\hat{f}^1(\sigma) - \sigma\| \leqslant 2\eta$ if $\sigma \in U_i$ for some i.

For each i, j and n choose simplices X_n^{ij} and Y_n^{ij} such that:

(4) X_n^{ij} is contained in the interior of Y_n^{ij}, which in turn is contained in the interior of U_i;
(5) σ_n^{ij} is the barycenter of both X_n^{ij} and Y_n^{ij};
(6) Y_n^{ij} is contained in the η-radius ball around σ^{ij};
(7) $\{Y_n^{ij}\}_{i,j}$ are pairwise disjoint for each n.

Each dimension $\dim(\Sigma_n) - 1$ face F_n^{ij} of X_n^{ij} is contained in a unique hyperplane $H_{F_n^{ij}}$ in $\mathbb{R}^{S_n}$ orthogonal to Σ_n. Denote by $H^*_{F_n^{ij}}$, $* \in \{+, -\}$, the positive and negative half-spaces defined by this hyperplance. For a fixed X_n^{ij}, if a hyperplane $H_{F_n^{ij}}$ intersects $\sigma_n^{\hat{i}\hat{j}}$ for $\hat{i} \neq i$ or $\hat{j} \neq j$, then the vertices of F_n^{ij} are not in general position in Σ_n, i.e., they belong to a subspace of Σ_n of dimension strictly lower than $\dim(\Sigma_n)$. Therefore, choosing the vertices of X_n^{ij} conveniently in the interior of Y_n^{ij}, we can assume that $H_{F_n^{ij}}$ does not intersect $\sigma_n^{\hat{i}\hat{j}}$ for $\hat{i} \neq i$ or $\hat{j} \neq j$. We assume this holds for each i, j, n. We define $\mathcal{X}_n$. Consider the intersection of the hyperplane $H_{F_n^{ij}}$ with Σ_n, for each i, j.

This defines a polyhedral subdivision where a cell of the subdivision is given by the intersection of finitely many half-spaces $H^*_{F^{ij}_n}$.[5] Due to our assumption, the carrier of σ^{ij}_n in this subdivision is a cell of dimension $\dim(\Sigma_n)$. By construction of this subdivision, X^{ij}_n is immediately a subcomplex of $\mathcal{X}_n$. This proves 3(e).

Use Step 1 to construct for each n and i, j an affine homeomorphism $f^{ij}_n : X^{ij}_n \to Y^{ij}_n$ satisfying property (1) of Lemma 4.2 and such that $f^{ij} = \prod_n f^{ij}_n$ satisfies property (2) of Lemma 4.2. Letting A be the affine space generated by $\Sigma - \Sigma$, define $d : (\Sigma \setminus \cup_i(U_i \setminus \partial U_i)) \cup (\cup_{i,j} X^{ij}) \to A$ by $d(\sigma) = \sigma - \hat{f}^1(\sigma)$ if $\sigma \notin \cup_i(U_i \setminus \partial U_i)$; $d(\sigma) = \sigma - f^{ij}(\sigma)$ if $\sigma \in X^{ij}$. The only zeros of d are the σ^{ij}'s. The set U_i is semi-algebraic and $U_i \setminus \partial U_i$ is an open and connected set. Therefore, $(U_i, \partial U_i)$ is a pseudomanifold with boundary. Hence, $U_i \setminus \cup_j(X^{ij} \setminus \partial X^{ij})$ is a pseudo-manifold with boundary $(\cup_j \partial X^{ij}) \cup \partial U_i$ and the restriction of d to this boundary has degree zero. As there are at least two players with at least two pure strategies each, the dimension of Σ is at least two. Therefore, by the Hopf Extension Theorem, d extends to the sets U_i in such a way that its norm over $\cup_i U_i$ is still no more than 2η and it has no additional zeros.[6] Thus, we have a displacement map defined over the whole of Σ, denoted still by d. Define f as $\mathrm{Id} - d$.

Outside $\cup_i U_i$, d is the displacement of $\hat{f}^1$, whose graph is contained in $\mathrm{Graph}(\mathrm{BR}_\varepsilon)$; on the U_i's d has norm 2η or less, which by the choice of η, therefore, means the graph of f is contained in $\mathrm{Graph}(\mathrm{BR}_\varepsilon)$, proving point (1) of the Lemma. Point 3(c) of the lemma follows from the fact that the diameter of Y^{ij} is at most 2η; all other points of the lemma are obvious from the construction of the function f. □

Step 3. From here on, we fix the map f given in Step 2, along with the multisimplices $X^{ij} \equiv \prod_n X^{ij}_n$ and $Y^{ij} \equiv \prod_n Y^{ij}_n$. We now describe a triangulation $\mathcal{T}_n$ of Σ_n for each n.

Lemma 4.3. *For each n, there exists a triangulation $\mathcal{T}_n$ of Σ_n such that:*

(1) $(\sigma, \tau) \in Graph(BR_\varepsilon)$ if for each n, τ_n is a vertex of $\mathcal{T}_n$ and $f_n(\sigma)$ belongs to the simplicial neighborhood of the closed star of τ_n;
(2) For each i, j, there exists a simplex $T^{ij}_n \subset X^{ij}_n$ of $\mathcal{T}_n$ with σ^{ij}_n as its barycenter;
(3) For each i, j, letting $T^{ij} \equiv \prod_n T^{ij}_n$, if $\sigma \notin \cup_{i,j}(T^{ij} \setminus \partial T^{ij})$, there exists n such that for each τ_n that belongs to the carrier of σ_n in $\mathcal{T}_n$, σ_n does not belong to the carrier of $f_n(\sigma_{-n}, \tau_n)$ in $\mathcal{T}_n$.

[5] In the Appendix, we elaborate on polyhedral subdivisions constructed through this procedure.

[6] The version of the Hopf Extension Theorem we use here is Corollary 18, Chapter 8 in [22]. We note this Corollary is stated for singular cohomology (with coefficients in $\mathbb{Z}$), and provides a condition for the extension of $d : (\cup_j \partial X^{ij}) \cup \partial U_i \to A - \{0\}$ to $U_i \setminus \cup_j(X^{ij} \setminus \partial X^{ij})$. The condition is that $\delta(d^*(s^*)) = 0$, where s^* is a generator of $H^m(A - \{0\})$ ($A - \{0\}$ being a homotopy sphere), where $m = \dim(A) - 1$ and δ is the coboundary cohomology morphism of the long exact sequence of the pair $(U_i \setminus \cup_j(X^{ij} \setminus \partial X^{ij}), (\cup_j \partial X^{ij}) \cup \partial U_i)$. If the degree of $d : (\cup_j \partial X^{ij}) \cup \partial U_i \to A - \{0\}$ is zero (because it is the displacement of $\hat{f}^1$), then by definition of the singular cohomology functor, $d^*(s^*) = 0$, which implies $\delta(d^*(s^*)) = 0$.

Proof. There exists $\tilde{\eta} > 0$ such that:

(4) if $\sigma \notin \cup_{i,j}(X^{ij} \setminus \partial X^{ij})$, then there exists n such that $\|f_n(\sigma_{-n}, \tau_n) - \sigma_n\| > \tilde{\eta}$, for all τ within $\tilde{\eta}$ of σ;

(5) $(\sigma, \tau) \in \text{Graph}(\text{BR}_\varepsilon)$, if τ is within $\tilde{\eta}$ of $f(\sigma)$.

Let T_n^{ij} be the simplex that has as its vertices $\{(1 - .5\tilde{\eta})\sigma_n^{ij} + .5\tilde{\eta}v \mid v \text{ a vertex of } X_n^{ij}\}$. Taking $\tilde{\eta} > 0$ smaller if necessary, we can assume that T_n^{ij} is contained in the interior of the carrier of σ_n^{ij} in $\mathcal{X}_n$ (cf. Lemma 4.2). Obviously T_n^{ij} is contained in X_n^{ij} for each n and i, j. Take a triangulation $\mathcal{T}_n'$ of Σ_n which refines $\mathcal{X}_n$ such that:

(6) T_n^{ij} is a simplex of $\mathcal{T}_n'$ for each i, j;

(7) The diameter of each simplex in $\mathcal{T}_n'$ is no more than $\tilde{\eta}$.

Then, properties (1) and (2) hold for the triangulation $\mathcal{T}_n'$ (from (6) and (7)); property (3) also holds for this triangulation, for $\sigma \notin \cup_{i,j}(X^{ij} \setminus \partial X^{ij})$. Since f_n^{ij} is an affine homeomorphism of X_n^{ij} onto Y_n^{ij}, both of which have σ_n^{ij} as their barycenters, we have $f_n^{ij}(\partial T_n^{ij}) \cap T_n^{ij} = \emptyset$. Therefore, there exists a sufficiently fine subdivision $\mathcal{T}_n$ of $\mathcal{T}_n'$ modulo $\cup_{i,j}T_n^{ij}$ —i.e., it subdivides all simplices that do not intersect the T_n^{ij}'s (cf. [24])—such that property (3) holds even for $\sigma \in \cup_{i,j}(X^{ij} \setminus (T^{ij} \setminus \partial T^{ij}))$ $\square$

Step 4. We now construct an equivalent game $\tilde{G}$ in polytope form. The best reply correspondence of this game is denoted $\tilde{BR}$ and we analogously denote by $\tilde{BR}_\varepsilon$ the ε-best-reply correspondence. For each n, let $\tilde{S}_{n,0}$ be the set of vertices of the triangulation $\mathcal{T}_n$ obtained in Step 3. Let $\tilde{\Sigma}_{n,0} \equiv \Delta(\tilde{S}_{n,0})$ and $\tilde{\Sigma}_{n,1} \equiv \Delta(\tilde{S}_{n,1})$, where $\tilde{S}_{n,1} = \tilde{S}_{n+1,0}$, with the convention that $n + 1 = 1$ if $n = N$. Let $\tilde{\Sigma}_n = \tilde{\Sigma}_{n,0} \times \tilde{\Sigma}_{n,1}$ and $\tilde{\Sigma} = \prod_n \tilde{\Sigma}_n$. Let $\phi_{n,0} : \tilde{\Sigma}_{n,0} \to \Sigma_n$ be the affine function that sends each $\tilde{s}_{n,0}$ to the corresponding mixed strategy $\sigma_n \in \Sigma_n$. We also use $\phi_{n,1} : \tilde{\Sigma}_{n,1} \to \Sigma_{n+1}$ to denote the corresponding map for the second factor, $\tilde{\Sigma}_{n,1}$. The payoffs are now defined as $\tilde{G}_n(\tilde{\sigma}) \equiv G_n((\phi_{n,0}(\tilde{\sigma}_{n,0})_{n \in \mathcal{N}})$. In particular, the choice of the strategy $\sigma_{n,1}$ of any player n is payoff-irrelevant. Obviously, $\tilde{G}$ is equivalent to G.

Let $\text{Graph}(\tilde{\text{BR}}_\varepsilon)$ be the graph of the ε-best-reply correspondence of $\tilde{G}$. We now use the map f to construct a map $\tilde{f} : \tilde{\Sigma} \to \tilde{\Sigma}$ such that for each n, $\tilde{f}_n$ is independent of n's coordinate $\tilde{\sigma}_n$. Let $\tilde{\beta}_{n,0} : \Sigma_n \to \tilde{\Sigma}_{n,0}$ be the map that assigns to each σ_n, the barycentric coordinates of σ_n in the triangulation $\mathcal{T}_n$: that is, σ_n can be written uniquely as a convex combination of the vertices of its carrier in $\mathcal{T}_n$ and $\tilde{\beta}_{n,0}$ assigns σ_n to the same convex combination in the face of $\tilde{\Sigma}_{n,0}$ generated by the vertices of that carrier. The map $\tilde{\beta}_{n,1} : \Sigma_{n+1} \to \tilde{\Sigma}_{n,1}$ is defined similarly. We denote

$$\tilde{\beta}_{n,k} \equiv \times_{\tilde{s}_{n,k}} \beta_{n,k}^{\tilde{s}_{n,k}}$$

where $\beta_{n,0}^{\tilde{s}_{n,0}} : \Sigma_n \to \mathbb{R}$ assigns the value of the coordinate $\tilde{s}_{n,0}$ and $\beta_{n,1}^{\tilde{s}_{n,1}} : \Sigma_{n+1} \to \mathbb{R}$ assigns the value of the coordinate $\tilde{s}_{n,1}$.

Define $\tilde{f}$ by $\tilde{f}_n(\tilde{\sigma}) = (\tilde{\tau}_{n,0}, \tilde{\tau}_{n,1})$, where:

$$\tilde{\tau}_{n,0} \equiv \tilde{\beta}_{n,0}(f_n(\phi_{-n,0}(\tilde{\sigma}_{-n,0}), \phi_{n-1,1}(\tilde{\sigma}_{n-1,1})))$$

with the convention that $n-1 = N$ if $n = 1$, and

$$\tilde{\tau}_{n,1} \equiv \tilde{\beta}_{n,1}(\phi_{n+1,0}(\tilde{\sigma}_{n+1,0})).$$

Let $\tilde{\Sigma}^*_{n,k} \equiv \tilde{\beta}_{n,k}(\Sigma_{n+k})$, for $k = 0, 1$. The set $\tilde{\Sigma}^*_{n,k}$ is then the space of a triangulation $\mathcal{S}^*_{n,k}$ given by a subcomplex of the face complex of $\Delta(\tilde{S}_{n,k})$. For each n and $k = 0, 1$, let $\tilde{T}^{ij}_{n,k} \equiv \tilde{\beta}_{n,k}((f^{ij}_{n+k})^{-1}(T^{ij}_{n+k}))$. Then $\tilde{T}^{ij}_{n,k}$ is contained in the face $\tilde{\beta}_{n,k}(T^{ij}_{n,k})$ of $\tilde{\Sigma}_{n,k}$.

Let

$$\tilde{\Sigma}^*_n \equiv \tilde{\Sigma}^*_{n,0} \times \tilde{\Sigma}^*_{n,1} \quad ; \quad \tilde{T}^{ij}_n \equiv \tilde{T}^{ij}_{n,0} \times \tilde{T}^{ij}_{n,1} \quad ; \quad \tilde{\Sigma}^* \equiv \prod_n \tilde{\Sigma}^*_n \quad ; \quad \tilde{T}^{ij} \equiv \prod_n \tilde{T}^{ij}_n.$$

For each i, j, n and k, let $\tilde{\sigma}^{ij}_{n,k} \equiv \tilde{\beta}_{n,k}(\sigma^{ij}_{n+k})$, $\tilde{\sigma}^{ij}_n \equiv (\tilde{\sigma}^{ij}_{n,0}, \tilde{\sigma}^{ij}_{n,1})$ and $\tilde{\sigma}^{ij} \equiv (\tilde{\sigma}^{ij}_n)_{n\in\mathcal{N}}$. By Property (2) of Lemma 4.3, for each n, i, j and $k = 0, 1$, $\tilde{\sigma}^{ij}_{n,k}$ is the barycenter of both $\tilde{T}^{ij}_n$ and $\tilde{\beta}_{n,k}(T^{ij}_{n,k})$.

The following lemma summarizes key properties of $\tilde{f}$.

Lemma 4.4. *The function $\tilde{f}$ maps $\tilde{\Sigma}$ into $\tilde{\Sigma}^*$ and player n's coordinate functions $\tilde{f}_{n,0}$ and $\tilde{f}_{n,1}$ are independent of the coordinate $\tilde{\sigma}_n$. Moreover,*

1. *$\tilde{f}_{n,1}$ is a function of $\tilde{\sigma}_{n+1,0}$ and the restriction of the function to $\tilde{\Sigma}^*_{n+1,0}$ is affine on each face of $\tilde{\Sigma}^*_{n+1,0}$;*
2. *the restriction of $\tilde{f}$ to each $\tilde{T}^{ij}$ decomposes into affine homeomorphisms $\tilde{f}_{n,0} : \tilde{T}^{ij}_{n-1,1} \to \tilde{\beta}_{n,0}(T^{ij}_n)$, $\tilde{f}_{n,1} : \tilde{T}^{ij}_{n+1,0} \to \tilde{\beta}_{n,1}(T^{ij}_{n+1})$;*
3. *if for each $n \in \mathcal{N}, k = 0, 1$, $\tilde{f}_{n,k}(\tilde{\sigma})$ belongs to the simplicial neighborhood of the closed star of a pure strategy $\tilde{s}_{n,k} \in \tilde{\Sigma}^*_{n,k}$ in the triangulation $\mathcal{S}^*_{n,k}$, then $(\tilde{\sigma}, \tilde{s}) \in Graph(\tilde{BR}_\varepsilon)$;*
4. *if $\tilde{\sigma} \notin \tilde{T}^{ij} \setminus \partial\tilde{T}^{ij}$ for any i, j and if for all n, $\phi_{n-1,1}(\tilde{\sigma}_{n-1,1})$ belongs to the carrier of $\phi_{n,0}(\tilde{\sigma}_{n,0})$ in $\mathcal{T}_n$, then there exists n such that $\tilde{\sigma}_{n,0}$ does not belong to the carrier of $\tilde{f}_{n,0}(\tilde{\sigma})$ in $\mathcal{S}^*_{n,0}$;*
5. *the only fixed points of $\tilde{f}$ are the profiles $\tilde{\sigma}^{ij}$ and each $\tilde{\sigma}^{ij}$ has index r^{ij}.*

Proof. It follows from the definition of $\tilde{f}$ that each player's coordinate functions are independent of his strategies. It is also easily checked that $\tilde{f}$ maps into $\tilde{\Sigma}^*$. The numbered items (1), and (2) are also immediate from the definition. As for (3), it follows from property (1) of Lemma 4.3. Property (4) follows from property (3) of Lemma 4.3 in the case where $(\phi_{n,0}(\tilde{\sigma}_{n,0}))_{n\in\mathcal{N}} \notin \cup_{i,j}(T^{ij} \setminus \partial T^{ij})$. Consider now the case where $(\phi_{n,0}(\tilde{\sigma}_{n,0}))_{n\in\mathcal{N}}$ belongs to the set $T^{ij} \setminus (\partial T^{ij} \cup ((f^{ij})^{-1}(T^{ij} \setminus \partial T^{ij}))$ for some i, j. There exists n such that $\phi_{n,0}(\tilde{\sigma}_{n,0})$ belongs to the set $T^{ij}_n \setminus (\partial T^{ij}_n \cup ((f^{ij}_n)^{-1}(T^{ij}_n \setminus \partial T^{ij}_n))$. As $\phi_{n-1,1}(\tilde{\sigma}_{n-1,1})$ belongs to the same simplex as $\phi_{n,0}(\tilde{\sigma}_{n,0})$, it lies in $T^{ij}_n \setminus ((f^{ij}_n)^{-1}(T^{ij}_n \setminus \partial T^{ij}_n))$. Its image under f_n^{ij} (recall it depends only on $\phi_{n-1,1}(\tilde{\sigma}_{n-1,1})$) lies outside $T_n^{ij} \setminus \partial T_n^{ij}$. This completes the proof of property (4).

There remains to prove point (5). The index of $\tilde{\sigma}^{ij}$ can be computed using the restriction of $\tilde{f}$ to $\tilde{T}^{ij}$. On this set, the maps decompose into homeomorphisms $\hat{f}_n \equiv \tilde{f}_{n,0} \times \tilde{f}_{n-1,1} : \tilde{T}_{n,0}^{ij} \times \tilde{T}_{n-1,1}^{ij} \to \tilde{\beta}_{n,0}(T_n^{ij}) \times \tilde{\beta}_{n-1,1}(T_n^{ij})$, where $\hat{f}_n(\tilde{\sigma}_{n,0}, \tilde{\sigma}_{n-1,1}) = (\tilde{\beta}_{n,0}(f_n^{ij}(\phi_{n-1,1}(\tilde{\sigma}_{n-1,1}))), \tilde{\sigma}_{n,0})$. For $\lambda \in [0,1]$, let $\hat{f}^\lambda$ be the map sending $(\tilde{\sigma}_{n,0}, \tilde{\sigma}_{n-1,1})$ to $(\tilde{\beta}_{n,0}(f_n^{ij}((1-\lambda)\phi_{n,0}(\tilde{\sigma}_{n,0}) + \lambda\phi_{n-1,1}(\tilde{\sigma}_{n-1,1}))), (1-\lambda)\tilde{\sigma}_{n,1} + \lambda\tilde{\sigma}_{n,0})$. When $\lambda = 1$, we get the map $\hat{f}^1$. For $\lambda > 0$, the unique fixed point is $(\tilde{\sigma}_n^{ij}, \tilde{\sigma}_{n-1,1}^{ij})$ and for $\lambda = 0$, we have a component of fixed points $\{ \tilde{\sigma}_{n,0}^{ij} \} \times \tilde{T}_{n,1}^{ij}$. By the multiplication property of the index, the index of the component of fixed points for $\lambda = 0$ is the index of σ_n^{ij} under the map f_n^{ij}. Therefore, the index of $\tilde{\sigma}^{ij}$ is r^{ij}, as claimed. $\square$

Step 5. We now construct a family of ε-perturbed games, indexed by $\tilde{\sigma} \in \tilde{\Sigma}$. First we need to introduce some notation. Given vectors $\tilde{g}^0 \in \prod_n \mathbb{R}^{\tilde{S}_{n0}}$ and $\tilde{g}^1 \in \prod_n \mathbb{R}^{\tilde{S}_{n,1}}$, we define the game $\tilde{G} \oplus \tilde{g}^0 \oplus \tilde{g}^1$ as a polytope-form game where for each $\tilde{\sigma}$ and each n, the payoff is $\tilde{G}_n(\tilde{\sigma}) + \tilde{g}_n^0 \cdot \tilde{\sigma}_{n,0} + \tilde{g}_n^1 \cdot \tilde{\sigma}_{n,1}$ (where $a \cdot b$ denotes the scalar product of the vectors a and b). Thus $\tilde{g}_{n,\tilde{s}_{n,0}}^0$ represents a "bonus" for playing $\tilde{s}_{n,0}$ and similarly $\tilde{g}_{n,\tilde{s}_{n,1}}^0$, for each $\tilde{s}_{n,1}$. Given functions $\tilde{g}^0 : \tilde{\Sigma} \to \prod_n \mathbb{R}_+^{\tilde{S}_{n,0}}$, $\tilde{g}^1 : \tilde{\Sigma} \to \prod_n \mathbb{R}_+^{\tilde{S}_{n,1}}$ and profile $\tilde{\sigma}$, we have a finite polytope-form game $\tilde{G} \oplus \tilde{g}^0(\tilde{\sigma}) \oplus \tilde{g}^1(\tilde{\sigma})$.

For the time being fix $\varepsilon_0 > 0$. The exact choice will be determined later. We first define $\tilde{g}^1$. For each n and $\tilde{s}_{n,1}$, it depends only on $\tilde{\sigma}_{n+1,0}$ and it is given by

$$\tilde{g}_{n,\tilde{s}_{n,1}}^1(\tilde{\sigma}_{-n}) = \varepsilon_0 \tilde{\beta}_{n,1}^{\tilde{s}_{n,1}}(\phi_{n+1,0}(\tilde{\sigma}_{n+1,0})).$$

where $\tilde{\beta}_{n,1}^{\tilde{s}_{n,1}}$ gives the $\tilde{s}_{n,1}$ coordinate of the barycentric coordinates w.r.t. the triangulation $\mathcal{T}_{n+1}$. To define $\tilde{g}^0$, recall that $\tilde{\Sigma}_{n,0}^*$ is a triangulation in $\tilde{\Sigma}_{n,0}$. Let $\tilde{\gamma}_{n,s_{n,0}} : \tilde{\Sigma}_{n,0}^* \to [0,1]$ be a Urysohn function that is one on the closed star of $\tilde{s}_{n,0}$ in the complex $\mathcal{S}_{n,0}^*$ and zero outside the simplicial neighborhood of the closed star of $\tilde{s}_{n,0}$ in this complex. Then,

$$\tilde{g}_{n,\tilde{s}_{n,0}}^0(\tilde{\sigma}_{-n}) \equiv \tilde{\gamma}_{n,s_{n,0}}(\tau_n)[r(\phi_{-n,0}(\sigma_{-n,0})) - G_n(\tilde{\sigma}_{-n,0}, \tilde{\phi}_{n,0}(s_{n,0}))] + \varepsilon_0 \tilde{f}_{n,\tilde{s}_{n,0}}(\tilde{\sigma}_{-n}),$$

where

$$\tau_n = \tilde{\beta}_{n,0}(f_n(\phi_{-n,0}(\tilde{\sigma}_{-n}), \phi_{n-1,1}(\tilde{\sigma}_{n-1,1}))),$$

and $r(\phi_{-n,0}(\sigma_{-n,0}))$ is the best payoff n can obtain against the profile $\phi_{-n,0}(\tilde{\sigma}_{-n,0})$ in G. The following lemma sets out key properties of the perturbations. Its proof follows immediately from the construction in this step, along with the properties of the map $\tilde{f}$ laid out in Lemma 4.4.

Lemma 4.5. *Each player n's coordinate functions $\tilde{g}_n^0$ and $\tilde{g}_n^1$ are independent of $\tilde{\sigma}_n$. Moreover:*

(1) the restriction of $\tilde{g}_{n,\tilde{s}_{n,1}}^1$ to $\Sigma_{n+1,0}^*$ is affine on each simplex of $\mathcal{S}_{n+1,0}^*$;

(2) the restriction of $\tilde{g}^0$ to each $\tilde{T}^{ij}$ decomposes into affine maps $\tilde{g}_{n,\tilde{s}_{n,0}}^{ij,0} : \tilde{T}_{n-1,1}^{ij} \to \mathbb{R}_+$;

(3) $\|\tilde{g}^0\| + \|\tilde{g}^1\| < \varepsilon$, *if* ε_0 *is small.*

We conclude this step with a lemma concerning best replies in these perturbed games. Its proof, too, is immediate from the construction of the perturbed games.

Lemma 4.6. *The profile* $(\tilde{s}_{n,0}, \tilde{s}_{n,1})$ *is a best reply to* $\tilde{\sigma}$ *in the game* $\tilde{G} \oplus \tilde{g}^0(\tilde{\sigma}) \oplus \tilde{g}^1(\tilde{\sigma})$ *only if:*

(1) $\tilde{s}_{n,0}$ *is a vertex of the carrier of* $\tilde{f}_{n,0}(\tilde{\sigma}_{-n,0}, \tilde{\sigma}_{n-1,1})$ *in* $\mathcal{S}^*_{n,0}$*;*

(2) $\tilde{s}_{n,1}$ *is a vertex of the carrier of* $\tilde{f}_{n,1}(\tilde{\sigma}_{n+1,0})$ *in* $\mathcal{S}^*_{n+1,0}$*.*

Step 6. Let $\tilde{\varphi} : \tilde{\Sigma} \twoheadrightarrow \tilde{\Sigma}$ be the correspondence that assigns to each $\tilde{\sigma}$ the set of best replies to it in the polytope-form game $\tilde{G} \oplus \tilde{g}^1(\tilde{\sigma}) \oplus \tilde{g}^2(\tilde{\sigma})$. The correspondence $\tilde{\varphi}$ is a non-empty, compact, and convex-valued. It is also upper semi-continuous. Hence it has a fixed point and a well-defined index for its fixed points. We now characterize the fixed points.

Lemma 4.7. *The fixed points of* $\tilde{\varphi}$ *are the profiles* $\tilde{\sigma}^{ij}$*. The index of each* $\tilde{\sigma}^{ij}$ *is* r^{ij}*.*

Proof. To prove this step we will show that for all $\lambda \in [0,1]$ the only fixed points of $\lambda\tilde{f} + (1-\lambda)\tilde{\varphi}$ are the $\tilde{\sigma}^{ij}$'s. The proof then follows from item (5) of Lemma 4.4.

Take some $\tilde{\sigma}$ that is a fixed point of $\lambda\tilde{f} + (1-\lambda)\tilde{\varphi}$ for some $\lambda \in [0,1]$. It follows from Lemma 4.6 that $\tilde{\sigma}_{n,0}$ belongs to the simplex of $\mathcal{S}^*_{n,0}$ that contains $\tilde{f}_{n,0}(\tilde{\sigma}_{-n,0}, \tilde{\sigma}_{n-1,1})$. An optimal $\tilde{s}_{n-1,1}$ has to be a vertex of the simplex containing $\tilde{\sigma}_{n,0}$ (which is also the carrier of the image of $\tilde{\sigma}_{n,0}$ under $\tilde{f}_{n-1,1}$). Hence $\tilde{\sigma}_{n-1,1}$ belongs to the simplex containg $\tilde{\sigma}_{n,0}$. By property (4) of Lemma 4.4, $\tilde{\sigma}$ must be in $\tilde{T}^{ij}$ for some i,j. It is easy to check that in this case $\tilde{\sigma} = \tilde{\sigma}^{ij}$. □

Step 7. In this step, we describe a triangulation of $\tilde{\Sigma}_n$ for each n that allows us to convert the continuous family of perturbed games to a single perturbed game. There exists $\xi > 0$ such that:

(A) $\tilde{\sigma} \notin \cup_{i,j}(\tilde{T}^{ij} \setminus \partial\tilde{T}^{ij})$ is not an equilibrium of $\tilde{G} \oplus \tilde{g}^0 \oplus \tilde{g}^1$ if $\|\tilde{g}^0 - \tilde{g}^0(\tilde{\sigma})\| + \|\tilde{g}^1 - \tilde{g}^1(\tilde{\sigma})\| < \xi$.

(B) If $\tilde{\sigma} \in \tilde{T}^{ij}$ with $\tilde{\sigma}_{n-1,1} \in \partial\tilde{T}^{ij}_{n-1,1}$ for some n and i,j, there is at least one vertex of $\tilde{T}^{ij}_{n-1,1}$ that is not a best reply in $\tilde{G} \oplus \tilde{g}^0 \oplus \tilde{g}^1$ against $\tilde{\sigma}$ for any $\tilde{g}^1 \in \mathbb{R}^{\tilde{S}_{n,1}}$ that is within ξ of $\tilde{g}^1(\tilde{\sigma})$.

Claim (A) follows from the fact that if $\sigma \notin \cup_{i,j}(\tilde{T}^{ij} \setminus \partial\tilde{T}^{ij})$, then σ is not an equilibrium of $\tilde{G} \oplus \tilde{g}^0(\tilde{\sigma}) \oplus \tilde{g}^1(\tilde{\sigma})$ (cf. Lemma 4.7). Therefore, for sufficiently small perturbations $\tilde{g}^0$ of $\tilde{g}^0(\tilde{\sigma})$ and $\tilde{g}^1$ of $\tilde{g}^1(\tilde{\sigma})$, $\tilde{\sigma}$ is also not an equilibrium of the finite game $\tilde{G} \oplus \tilde{g}^0 \oplus \tilde{g}^1$.

For Claim (B), note that a best reply $\tilde{\tau}_{n-1,1}$ of player $n-1$ in $\tilde{G} \oplus \tilde{g}^0(\tilde{\sigma}) \oplus \tilde{g}^1(\tilde{\sigma})$ must be a vertex of the carrier of $\tilde{\sigma}_{n,0}$. If $\tilde{\sigma}_{n-1,1}$ is not a best reply to $\tilde{\sigma}$, then the result is immediate, since it puts positive probability on some vertex of $\tilde{T}^{ij}_{n-1,1}$ that is an inferior reply. Now, if $\tilde{\sigma}_{n-1,1}$ is a best reply to $\tilde{\sigma}$, then both $\tilde{\sigma}_{n,0}$ and $\tilde{\sigma}_{n-1,1}$ are located in the boundary of $\tilde{T}^{ij}_{n-1,1}$, and again some vertex of $\tilde{T}^{ij}_{n-1,1}$ is not a best reply against $\tilde{\sigma}_{n,0}$. Evidently, for any sufficiently small perturbation $\tilde{g}^1$ of $\tilde{g}^1(\tilde{\sigma})$, the same is true.

We can choose, in addition, ξ small such that any ξ-perturbation of $\tilde{g}(\cdot)$ yields an ε-perturbation of the payoffs in $\tilde{G}$. Since $g(\tilde{\sigma})$ is uniformly continuous, there exists $\zeta > 0$ such that $\|g(\tilde{\sigma}) - g(\tilde{\sigma}')\| < \xi$ if $\|\tilde{\sigma} - \tilde{\sigma}'\| < \zeta$. Recall that $\tilde{S}_{n,0}$ is the set of vertices of $\mathcal{T}_n$ and $\tilde{\Sigma}_{n,0} = \tilde{\Sigma}_{n-1,1} = \Delta(\tilde{S}_{n,0})$. Therefore, any triangulation of $\Delta(\tilde{S}_{n,0})$ induces a corresponding triangulation of $\tilde{\Sigma}_{n,0}$ and $\tilde{\Sigma}_{n-1,1}$.

Lemma 4.8. *For each n, there exists a triangulation $\tilde{\mathcal{T}}_n$ of $\Delta(\tilde{S}_{n,0})$ with the following properties.*

(1) the diameter of each simplex is less than ζ;
(2) for each i, j, $\tilde{T}^{ij}_{n,0}$ is the space of a subcomplex of $\mathcal{T}_n$ and $\sigma^{ij}_{n,0}$ is in a generic position in the interior of a simplex $\hat{T}^{ij}_{n,0}$ of dimension $\dim(\Sigma_n)$ in the sense that $(\tilde{\sigma}^{ij}_{n,0}, \tilde{\sigma}^{ij}_{n,1})$ is not in the convex hull of $((dim)(\Sigma_n) + (dim)(\Sigma_{n+1}))$ vertices of $\hat{T}^{ij}_{n,0} \times \hat{T}^{ij}_{n,1}$;
(3) there is a convex piecewise linear function $\gamma_n : \Delta(\tilde{S}_{n,0}) \to \mathbb{R}_+$ such that:
  (a) the map is linear precisely on the simplices of $\tilde{\mathcal{T}}_n$;
  (b) the function is constant over each $\hat{T}^{ij}_{n,0}$.

Proof. See Appendix. □

Step 8. We now define a game $\hat{G}$ in normal form. Let $\hat{S}_{n,0}$ be the set of vertices of $\tilde{\mathcal{T}}_n$. For each n, let $\hat{S}_{n,1} \equiv \hat{S}_{n+1,0}$ and $\hat{S}_n \equiv \hat{S}_{n,0} \times \hat{S}_{n,1}$. The pure strategy set of player n is $\hat{G}$ is $\hat{S}_n$. Let $\hat{\Sigma}_{n,0} = \Delta(\hat{S}_{n,0})$ and $\hat{\Sigma}_{n,1} = \Delta(\hat{S}_{n,1})$. For $k = 0, 1$, let $\hat{\mu}_{n,k} : \hat{\Sigma}_n \to \hat{\Sigma}_{n,k}$ be the map that sends each $\hat{\sigma}_n$ to the marginal distribution over $\hat{S}_{n,k}$. Each $\hat{s}_{n,0} \in \hat{S}_{n,0}$ corresponds to a mixed strategy in $\tilde{\Sigma}_{n,0}$ Therefore, there exists for $k = 0, 1$, an affine map $\hat{\phi}_{n,k} : \hat{\Sigma}_{n,k} \to \tilde{\Sigma}_{n,k}$. Let $\hat{\phi}_n \equiv \hat{\phi}_{n,0} \times \hat{\phi}_{n,1}$ and $\hat{\mu}_n \equiv \hat{\mu}_{n,0} \times \hat{\mu}_{n,1}$. The payoffs in $\hat{G}$ are defined, for each player n, by $\hat{G}_n(\hat{\sigma}) = \tilde{G}_n(\hat{\phi}_0 \circ \hat{\mu}_0(\hat{\sigma}))$, where $\hat{\phi}_0 \equiv \times_n \hat{\phi}_{n,0}$ and $\hat{\mu}_0 \equiv \times_n \hat{\mu}_{n,0}$. Obviously $\hat{G}$ is equivalent to $\tilde{G}$ and hence to G.

For each n, i, j and $k = 0, 1$, let $\hat{S}^{ij}_{n,k}$ be the set of vertices of $\hat{T}^{ij}_{n+k,0}$ and let $\hat{\Sigma}^{ij}_{n,k}$ be the face of $\hat{\Sigma}_{n,k}$ consisting of strategies with support contained in $\hat{S}^{ij}_{n,k}$. Let $\hat{\Sigma}^{ij}_n$ be the set of mixed strategies $\hat{\sigma}_n$ such that $\hat{\mu}_{n,k}(\hat{\sigma}_n) \in \hat{\Sigma}^{ij}_{n,k}$ for each k. Let $\hat{\Sigma}^{ij} \equiv \prod_n \hat{\Sigma}^{ij}_n$. Let $\hat{C}^{ij}$ be the set of $\hat{\sigma} \in \hat{\Sigma}^{ij}$ such that $\hat{\phi}_{n,k}(\hat{\mu}_{n,k}(\hat{\sigma}_n))) = \tilde{\sigma}^{ij}_n$. The set $\hat{C}^{ij}$ is then a polytope, since $\hat{\phi}_{n,k} \circ \hat{\mu}_{n,k}$ is affine. By Property (2) of Lemma 4.8, each extreme point of $\hat{C}^{ij}_n$ belongs to a face of dimension $\dim(\Sigma_n) + \dim(\Sigma_{n+1})$. Let $\hat{\sigma}^{ij}$ be one of these extreme points of this polytope. For each n, let $\hat{S}^{ij,*}_n$ be the support of $\hat{\sigma}^{ij}$.

We will now construct an ε-perturbation of $\hat{G}$. In the next and final step, we show that the equilibria of this perturbed game are the points $\hat{\sigma}^{ij}$ and that the index of each $\hat{\sigma}^{ij}$ is r^{ij}.

The perturbation of the payoffs of $\hat{G}$ for each n has five components: $\hat{G}^0_n : \hat{\Sigma} \to \mathbb{R}_{++}$; $\hat{G}^1_n : \hat{\Sigma} \to \mathbb{R}_+$; $\hat{G}^*_n : \hat{\Sigma} \to \mathbb{R}_-$; $\hat{g}^0_n : \hat{S}_{n,0} \to \mathbb{R}$; and $\hat{g}^1_n : \hat{S}_{n,1} \to \mathbb{R}$. For $\hat{s} \in \hat{S}$, with $\hat{s}_m = (\hat{s}_{m,0}, \hat{s}_{m,1})$ for each m:

$$\begin{array}{rcl} \hat{G}^0_n(\hat{s}) & = & \hat{\phi}_{n,0}(\hat{s}_{n,0}) \cdot \tilde{g}^0_n(\hat{\phi}_{-n,0}(\hat{s}_{-n,0}), \hat{\phi}_{n-1,1}(\hat{s}_{n-1,1})) \\ \hat{G}^1_n(\hat{s}) & = & \hat{\phi}_{n,1}(\hat{s}_{n,1}) \cdot \tilde{g}^1_n(\hat{\phi}_{n+1,0}(\hat{s}_{n+1,0})) \\ \hat{G}^*_n(\hat{s}) & = & -\mathbb{1}_{[\cup_{i,j}(\hat{S}^{ij}_n \setminus \hat{S}^{ij,*}_n) \times \hat{S}^{ij}_{-n}]} \\ \hat{g}^0_n(\hat{s}_n) & = & -\gamma_n(\hat{\phi}_{n,0}(\hat{s}_{n,0})) \\ \hat{g}^1_n(\hat{s}_n) & = & -\gamma_{n+1}(\hat{\phi}_{n,1}(\hat{s}_{n,1})). \end{array}$$

The function $\hat{G}^0_n$ depends on $\hat{\sigma}_{-n}$ and $\hat{\sigma}_{n,0}$ (the marginal of strategies $\hat{\sigma}_n$ on $\hat{\Sigma}_{n,0}$); the function $\hat{G}^1$ depends on $\hat{\sigma}_{n+1,0}$ and $\hat{\sigma}_{n,1}$. When n's opponents play some $\hat{\sigma}_{-n}$ s.t. $\hat{\phi}_{-n}(\hat{\mu}_{-n}(\hat{\sigma}_{-n})) = \tilde{\sigma}_{-n} \in \tilde{T}^{ij}_{-n}$, then $\hat{G}^0_n(\hat{s}_n, \hat{\sigma}_{-n}) = \hat{\phi}_{n,0}(\hat{s}_{n,0}) \cdot \tilde{g}^0_n(\tilde{\sigma}_{-n,0}, \tilde{\sigma}_{n-1,1})$, i.e., $\hat{G}^0_n$ is affine over profiles $\hat{\sigma}_{-n}$ that project to $\tilde{T}^{ij}_{-n}$ (cf. item (2), Lemma 4.5). The function $\hat{G}^1_n$ is affine on subsets of $\hat{\Sigma}_{n+1,0}$ that project under $\phi_{n+1,0}$ to simplices of $\tilde{\mathcal{T}}_{n+1}$: recall from item (1) in Lemma 4.5 that this is the case in the subcomplex $\mathcal{S}^*_{n+1,0}$, and everywhere else $\tilde{g}^1_{n,\tilde{s}_{n,1}}$ is identically 0. The function $\hat{G}^*$ penalizes n's use of a strategy in $\cup_{i,j}\hat{S}^{ij}_n$ that does not belong to the support of $\hat{\sigma}^{ij}_n$ when n's opponents are playing a strategy in $\hat{S}^{ij}_{-n}$. The functions $\hat{g}^0_n$, $\hat{g}^1_n$ are bonus functions like we constructed for the game $\tilde{G}$.

For positive constants α and α^*, define the game $\hat{G}^{\alpha,\alpha^*} \equiv \hat{G} \oplus \hat{G}^0 \oplus \hat{G}^1 \oplus \alpha^*\hat{G}^* \oplus \alpha\hat{g}^0 \oplus \alpha\hat{g}^1$. Fix α such that $G^{\alpha,0}$ is an ε-perturbation. If α^* is chosen small as well then G^{α,α^*} is also an ε-perturbation of $\hat{G}$.

Step 9. Let us first analyze the game $\hat{G}^{\alpha,0}$, i.e, the perturbation with $\alpha^* = 0$. Obviously every profile that belongs to $\hat{C}^{ij}$ for some ij is an equilibrium of $\hat{G}^{\alpha,0}$. We will now show that these are the only equilibria. Let $\hat{\sigma}$ be an equilibrium of $\hat{G}^{\alpha,0}$. For each n, let $\tilde{\sigma}_{n,k} = \hat{\phi}_{n,k}(\hat{\mu}_{n,k}(\hat{\sigma}_n))$ for $k = 0, 1$. We will show that $\tilde{\sigma} = \tilde{\sigma}^{ij}$ for some i, j.

Any strategy $\hat{\tau}_n$ of player n in $\hat{G}^{\alpha,0}$ such that $\tilde{\sigma}_{n,k} = \hat{\phi}_{n,k}(\hat{\mu}_{n,k}(\hat{\tau}_n))$ for each k yields the same payoff in $\hat{G}$, $\hat{G}^0$, and $\hat{G}^1$. The choice of which among these strategies is optimal depends on the payoffs they obtain under $\hat{g}^k_n$ for $k = 0, 1$. It is now easy to see that the optimality of $\hat{\sigma}_n$ implies that $\hat{\sigma}_{n,0}$ is a mixture over the vertices of the simplex of $\tilde{\mathcal{T}}_n$ that contains $\tilde{\sigma}_{n-1,1}$ in its interior and $\hat{\sigma}_{n,1}$ is a mixture over the vertices of the simplex of $\tilde{\mathcal{T}}_{n+1}$ that contains $\tilde{\sigma}_{n+1,0}$ in its interior. Thus $\tilde{\sigma}$ is an equilibrium of the game $\tilde{G} \oplus g^0 \oplus g^1$ where:

$$\begin{array}{rcl} g^0_{n,\tilde{s}_{n,0}} & = & \sum_{\tilde{s}_{-n}} \prod_{m \neq n} \tilde{\sigma}_m(\tilde{s}_m) \tilde{g}^0_{n,\tilde{s}_{n,0}}(\tilde{s}_{-n,0}, \tilde{s}_{n-1,1}) \\ g^1_{n,\tilde{s}_{n,1}} & = & \sum_{\tilde{s}_{n+1}} \tilde{\sigma}_{n+1}(\tilde{s}_{n+1}) \tilde{g}^1_{n,\tilde{s}_{n,1}}(\tilde{s}_{n+1,0}). \end{array}$$

Since $\|g^k - \tilde{g}^k(\tilde{\sigma})\| < \xi$, it implies that $\tilde{\sigma}$ is contained in the interior of $\tilde{T}^{ij}$ for some i, j. On this set, $\hat{G}^0$ and $\hat{G}^1$ are affine, so that $g^k = \tilde{g}^k(\tilde{\sigma})$ for $k = 0, 1$. If $\tilde{\sigma} \neq \tilde{\sigma}^{ij}$, there is some n for which either (a) $\tilde{\sigma}_{n,0} \neq \tilde{\sigma}^{ij}_{n,0}$ or (b) $\tilde{\sigma}_{n,1} \neq \tilde{\sigma}^{ij}_{n,1}$. Case (a) implies that $\tilde{\sigma}_{n-1,1}$ belongs to the boundary of $\tilde{\mathcal{T}}^{ij}_{n-1,1}$. But by the choice of ξ and α, one of the vertices of $\tilde{T}^{ij}_{n,0}$ is not optimal (cf. (B) in Step 7), hence $\tilde{\sigma}_{n,0}$ cannot be optimal; in the case of (b), $\tilde{\sigma}_{n+1,0}$ cannot belong to the interior of $\tilde{T}^{ij}_{n+1,0}$ (by construction of g^1), which is impossible as $\tilde{\sigma}$ must be in the interior of $\tilde{T}^{ij}$. Therefore, $\sigma = \tilde{\sigma}^{ij}$ and $\hat{\sigma}$ belongs to $\hat{C}^{ij}$.

Observe that against any equilibrium $\hat{\sigma}$ of $\hat{G}^{\alpha,0}$ that projects to $\tilde{\sigma}^{ij}$, any strategy $(\hat{s}_{n,0}, \hat{s}_{n,1})$ such that either $\hat{s}_{n,k}$ is not a vertex of $\hat{T}^{ij}_{n+k}$ for some k is a strictly inferior reply to the equilibrium. Therefore, for any small α^*, any equilibrium $\hat{\sigma}^{\alpha^*}$ must map under $\hat{\phi} \circ \hat{\mu}$ to a $\tilde{\sigma}$ that belongs to some $\hat{T}^{ij}$ and must be close to $\hat{C}^{ij}$. The perturbed game being a polymatrix game, the set of equilibria is a finite union of polytopes. For any extreme point of such a polytope there is a point in the face of $\hat{\Sigma}^{ij}$ containing $\hat{\sigma}^{ij}$ that has the same marginals $\hat{\mu}_{n,k}(\hat{\sigma}^{ij}_n)$, for each n and $k = 0, 1$. Clearly, such a strategy dominates any other profile inducing the same marginals. Hence, there is a unique equilibrium for $\hat{G}^{\alpha,\alpha^*}$. As $\hat{\sigma}^{ij}$ is an equilibrium it is the unique one.

To finish the proof, we have to show that the index of $\hat{\sigma}^{ij}$ is r^{ij}. For that it is sufficient to show that the index of $\hat{C}^{ij}$ is r^{ij}. In a neighborhood of $\hat{\sigma}^{ij}$, all strategies not in $\hat{\Sigma}^{ij}_n$ are inferior for n. Deleting them results in a game where the bonus functions yield the same payoffs and we could set $\alpha = 0$. But now, given that $\hat{G}^0$ and $\hat{G}^1$ are affine, the index of $\hat{\sigma}^{ij}$ is the same as the index of $\tilde{\sigma}^{ij}$ under the correspondence φ.

Appendix: Proof of Lemma 4.8

Let $m \equiv \dim(\Delta(\tilde{S}_{n,0}))$, so the number of vertices of $\Delta(\tilde{S}_{n,0})$ is $m + 1$. Recall the definition of $\mathcal{T}_n$ in Lemma 4.3. Note that $\mathcal{T}_n$ can be assumed to have any finite number of vertices. Letting $K_n \equiv \#\{\tilde{\sigma}^{ij}_n\}_{i,j}$, we will therefore assume that $K_n[\dim(\Sigma_n) + 1] < m + 1$.

A polyhedral subdivision $\mathcal{P}_n$ of $\Delta(\tilde{S}_{n,0})$ is said to be *regular* (cf. [2]) if there exists a *height function* $h : P_n \to \mathbb{R}$ from the vertex set P_n of $\mathcal{P}_n$, where the vertex set $\{p_t\}_t$ of a maximal-dimensional cell of the subdivision is mapped to points in $\mathbb{R}$ s.t. $\{(p_t, h(p_t))\}_t$ spans a (non-vertical) hyperplane in $\Delta(\tilde{S}_{n,0}) \times \mathbb{R}$ and all other points $p \in P_n$ are such that $(p, h(p))$ is strictly above the hyperplane. In this case, we say that the set $\{p_t\}_t$ is *lifted* by h *to a hyperplane*, with all other vertices *lifted above* the hyperplane.

Let $\tilde{\mathcal{T}}_n$ be a regular triangulation with vertex set T_n. For each $p \in T_n$, let $v_p \in \Delta(\tilde{S}_{n,0})$ be any point with the same carrier as p in the face complex of $\Delta(\tilde{S}_{n,0})$. For $\varepsilon \geq 0$, let $p(\varepsilon) \equiv (1-\varepsilon)p + \varepsilon v_p$. For $\varepsilon > 0$ sufficiently small, define the triangulation $\mathcal{T}^{\varepsilon}_n$ as follows: $B(\varepsilon) \equiv [p_0(\varepsilon), ..., p_m(\varepsilon)]$ is a maximal dimensional cell of $\tilde{\mathcal{T}}^{\varepsilon}_n$, given by the convex hull of $\{p_0(\varepsilon), ..., p_m(\varepsilon)\}$ iff $B \equiv [p_0, ..., p_m]$ is a maximal dimensional cell in $\tilde{\mathcal{T}}_n$ given by the convex hull of $\{p_0, ..., p_m\}$.

Lemma 4.9. *Suppose $\tilde{\mathcal{T}}_n$ is a regular triangulation. There exists $\varepsilon > 0$ sufficiently small such that any triangulation $\tilde{\mathcal{T}}^{\varepsilon}_n$ is regular.*

Proof. Take $\{p_0, ..., p_m\}$ to be the vertex set of a maximal dimensional cell B of $\tilde{\mathcal{T}}_n$ and let $p \in T_n \backslash B$. Since $\tilde{\mathcal{T}}_n$ is regular, let its associated height function be denoted by h. Then $(p, h(p))$ lies above the hyperplane defined by $(p_t, h(p_t))_{t=0}^m$. Therefore, substituting p_t for $p_t(\varepsilon)$ and letting $\varepsilon > 0$ be sufficiently small, we have that $(p_t(\varepsilon), h(p_t(\varepsilon)))_{t=0}^m$ defines a non-vertical hyperplane in $\mathbb{R}^{m+2}$ with $(p(\varepsilon), h(p(\varepsilon))$ above this hyperplane, for any other vertex $p(\varepsilon)$. From this, we can define the graph of a convex piecewise linear function $h^\varepsilon : \Delta(\tilde{S}_{n,0}) \to \mathbb{R}$, by letting $h^\varepsilon(p) \equiv h(p)$ for every vertex p of $\mathcal{T}_n^\varepsilon$, and, for another arbitrary point p, we consider the carrier of p in $\tilde{\mathcal{T}}_n^\varepsilon$ and define h^ε by linear interpolation. □

We now define a *generalized barycentric subdivision* of the simplex $\Delta(\tilde{S}_{n,0})$, which is a minor generalization of the classic barycentric subdivision. Let $\tilde{\mathcal{T}}_n^0$ be the face complex of $\Delta(\tilde{S}_{n,0})$. Assume $\tilde{\mathcal{T}}_n^{k-1}$ is defined and let us define $\tilde{\mathcal{T}}_n^k$: the 0-*dimensional generalized barycenters* are the vertices of $\tilde{\mathcal{T}}_n^{k-1}$. For each 1-dimensional cell $\tilde{\tau}$ of $\tilde{\mathcal{T}}_n^{k-1}$, chose exactly one point b^1 in its interior. The point b^1 is referred to as a 1-*dimensional generalized barycenter* of $\tilde{\tau}$. One then proceeds by choosing points in the interior of cells of increasing dimension, in the exact same fashion as with the classical barycentric subdivision.

The full-dimensional cells of $\tilde{\mathcal{T}}_n^k$ are also defined in the exact same fashion as in the k-th-iterate of the classical barycentric subdivision: an m-dimensional cell of $\tilde{\mathcal{T}}_n^k$ is denoted by its vertices $(b_k^0, b_k^1, ..., b_k^m)$, where b_k^0 is a vertex of $\tilde{\mathcal{T}}_n^{k-1}$, b_k^1 a generalized barycenter of a 1-dimensional cell of $\tilde{\mathcal{T}}_n^{k-1}$, b_k^2 a generalized barycenter of a 2-dimensional cell of $\tilde{\mathcal{T}}_n^{k-1}$, etc. When generalized barycenters are chosen sufficiently near the actual barycenters, the diameter of a generalized barycentric subdivision shrinks, as k increases, just like with the classical barycentric subdivision.

Letting $\zeta > 0$ be as in the statement of Lemma 4.8, there exists $\delta > 0$ sufficiently small and $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, any choice of generalized barycenters within δ of the actual barycenters yields a generalized barycentric subdivision $\tilde{\mathcal{T}}_n^k$ with diameter less than ζ. Moreover, we can assume without loss of generality that for each i, j, the carrier of $\tilde{\sigma}_n^{ij}$ in $\tilde{\mathcal{T}}_n^k$ has all its vertices in the interior of $\tilde{T}_n^{ij}$. We fix from now on such a $\delta > 0$ and k_0.

We now define a polyhedral refinement of $\tilde{\mathcal{T}}_n^k$, which we call the *Eaves-Lemke (EL)-refinement* $\mathcal{P}_n$ *of* $\tilde{\mathcal{T}}_n^k$. This is precisely the same polyhedral refinement as the one used in [4], we recall its construction for completeness. For each $(m-1)$-dimensional cell τ of $\tilde{\mathcal{T}}_n^k$, let $H_\tau \equiv \{x \in \mathbb{R}^{m+1} \mid a_\tau \cdot x = b_\tau\}$ be the hyperplane that includes τ and is orthogonal to $\Delta(\tilde{S}_{n,0})$. Each full-dimensional cell of $\mathcal{P}_n$ is the intersection of $\Delta(\tilde{S}_{n,0})$ with a polyhedron $\cap_\tau H_\tau^i$, where $i \in \{+, -\}$ and H_τ^i is one of the two half-spaces defined by the hyperplane H_τ.

Lemma 4.10. *There exists a polyhedral subdivision $\mathcal{P}_n$ of $\Delta(\tilde{S}_{n,0})$ satisfying the following properties:*

(1) for each i, j, $\tilde{\sigma}_n^{ij}$ has a carrier ϱ^{ij} in $\mathcal{P}_n$ of dimension $\dim(\Sigma_n)$;
(2) there is a convex piecewise linear function $\gamma_n : \Delta(\tilde{S}_{n,0}) \to [0, 1]$ which is linear precisely in each cell of $\mathcal{P}_n$;
(3) the diameter of $\mathcal{P}_n$ is less than ζ.

Proof. We first focus on constructing a polyhedral subdivision that satisfies (1). In $\tilde{\mathcal{T}}_n^0$, choose generalized barycenters which are distinct from $\tilde{\sigma}_n^{ij}$, for each i, j, defining $\tilde{\mathcal{T}}_n^1$. Consider now the derived (EL)-polyhedral subdivision $\mathcal{P}_n^1$ of $\tilde{\mathcal{T}}_n^1$. If there exists i, j such that the carrier of $\tilde{\sigma}_n^{ij}$ in $\mathcal{P}_n^1$ has dimension less than $\dim(\Sigma_n)$, then there exists a hyperplance H_τ which contains $\tilde{\sigma}_n^{ij}$, where τ is a face containing an m-dimensional generalized barycenter b_1^m. Let $\{p_0, p_1, ..., p_{m-2}, b_1^m\}$ be affinely independent vertices of the triangulation $\tilde{\mathcal{T}}_n^1$ defining $H_\tau \cap \Delta(\tilde{S}_{n,0})$. If H_τ contains in addition $\tilde{\sigma}_n^{ij}$, then $\{p_0, p_1, ..., p_{m-2}, b_1^m, \tilde{\sigma}_n^{ij}\}$ is an affinely dependent set. Choosing therefore b_1^m outside an affine set with dimension strictly lower than m in $\Delta(\tilde{S}_{n,0})$ implies H_τ does not contain $\tilde{\sigma}_n^{ij}$. This procedure can be iteratively applied for each point $\tilde{\sigma}_n^{ij}$, ensuring all of them have a carrier of dimension $\dim(\Sigma_n)$ in $\mathcal{P}_n^1$. Suppose now we have obtained $\mathcal{P}_n^{k_0-1}$ from $\tilde{\mathcal{T}}_n^{k_0-1}$ satisfying the property that all $\tilde{\sigma}_n^{ij}$ have carriers with dimension equal to $\dim(\Sigma_n)$. Choose generalized barycenters in $\tilde{\mathcal{T}}_n^{k_0-1}$ such that each generalized barycenter is distinct from each $\tilde{\sigma}_n^{ij}$ and consider the (EL)-polyhedral subdivision $\mathcal{P}_n^{k_0}$. If there exists i, j, such that the carrier of $\tilde{\sigma}_n^{ij}$ in $\mathcal{P}_n^{k_0}$ has dimension less than $\dim(\Sigma_n)$, then it follows that there exists τ containing an m-dimensional generalized barycenter $b_{k_0}^m$ such that H_τ contains $\tilde{\sigma}_n^{ij}$. Similarly as before, let $\{p_0, p_1, ..., p_{m-2}, b_{k_0}^m\}$ be affinely independent vertices of the triangulation $\tilde{\mathcal{T}}_n^{k_0}$ defining $H_\tau \cap \Delta(\tilde{S}_{n,0})$. If H_τ contains in addition $\tilde{\sigma}_n^{ij}$, then $\{p_0, p_1, ..., p_{m-2}, b_{k_0}^m, \tilde{\sigma}_n^{ij}\}$ is an affinely dependent set. Choosing therefore $b_{k_0}^m$ outside an affine set with dimension strictly lower than m in $\Delta(\tilde{S}_{n,0})$ implies H_τ does not contain $\tilde{\sigma}_n^{ij}$. This procedure can be iteratively applied for each point $\tilde{\sigma}_n^{ij}$, ensuring all of them have a carrier of dimension $\dim(\Sigma_n)$ in $\mathcal{P}_n^{k_0}$. This shows $\mathcal{P}_n^{k_0}$ satisfies (1). Since $\mathcal{P}_n^{k_0}$ refines $\tilde{\mathcal{T}}_n^{k_0}$, its diameter is also less than ζ, implying (3). To show (2), consider the following function: $\gamma_n : \Delta(\tilde{S}_{n,0}) \to [0, 1]$, $\gamma_n(\sigma_n) \equiv \alpha \sum_\tau |a_\tau \cdot \sigma_n - b_\tau|$, where the scaling factor $\alpha > 0$ is chosen so as to let $\gamma_n(\Delta(\tilde{S}_{n,0})) \subseteq [0, 1]$. The function γ_n then satisfies (2). □

We now derive a triangulation from $\mathcal{P}_n$ in Lemma 4.10, by triangulating each full-dimensional cell of $\mathcal{P}_n$. This triangulation will add no vertices beyond those of $\mathcal{P}_n$, i.e., it will simply subdivide each m-dimensional cell of $\mathcal{P}_n$ into m-dimensional simplices, without adding new vertices to the triangulation. This triangulation, up to an arbitrarily small perturbation of its vertices, achieves the objectives of Lemma 4.8.

Lemma 4.11. *There exists a triangulation $\tilde{\mathcal{T}}_n$, which refines $\mathcal{P}_n$ without adding new vertices, satisfying the following conditions:*

(1) The diameter of $\tilde{\mathcal{T}}_n$ is less than ζ;
(2) For i, j, $\tilde{T}_n^{ij}$ is the space of a subcomplex of $\tilde{\mathcal{T}}_n$;
(3) For i, j, the carrier $\hat{T}_n^{ij}$ of $\tilde{\sigma}_n^{ij}$ in $\tilde{\mathcal{T}}_n$ has dimension $\dim(\Sigma_n)$;
(4) There is a convex piecewise linear function $\gamma_n : \Delta(\tilde{S}_{n,0}) \to \mathbb{R}_+$ such that:
   (4.1) γ_n is linear precisely on the simplices of $\tilde{\mathcal{T}}_n$;

(4.2) γ_n *is constant in each* $\hat{T}_n^{ij}$*;*

(5) For each i, j, $(\tilde{\sigma}_n^{ij}, \tilde{\sigma}_{n+1}^{ij})$ *is not in the convex hull of* $\dim(\Sigma_n) + \dim(\Sigma_{n+1})$ *(or less) vertices of* $\hat{T}_n^{ij} \times \hat{T}_{n+1}^{ij}$.

Proof. The polyhedral complex $\mathcal{P}_n$ refines the generalized barycentric subdivision $\tilde{\mathcal{T}}_n^{k_0}$. Therefore it also has diameter less than ζ and has $\tilde{T}_n^{ij}$ as the space of a subcomplex. There exists a piecewise-linear, convex function $\hat{\gamma}_n : \Delta(\tilde{S}_{n,0}) \to \mathbb{R}_+$, which is linear precisely in the cells of $\mathcal{P}_n$ (cf. Lemma 4.10). The map $\hat{\gamma}_n(\cdot)$ is in particular a height function in the set of vertices of $\tilde{\mathcal{T}}_n^k$. By Lemma 2.3.15 in [2], any sufficiently small (generic) perturbation γ_n' of this height function gives a refinement $\tilde{\mathcal{T}}_n$ of $\mathcal{P}_n$ (without adding vertices to $\mathcal{P}_n$), where $\tilde{\mathcal{T}}_n$ is a triangulation, yielding a convex piecewise linear function $\gamma_n' : \Delta(\tilde{S}_{n,0}) \to \mathbb{R}_+$ which is linear precisely in the cells of $\tilde{\mathcal{T}}_n$.

We show that it is without loss of generality to assume that the carrier of $\tilde{\sigma}_n^{ij}$ in $\tilde{\mathcal{T}}_n$ is a full-dimensional cell of $\tilde{\mathcal{T}}_n$. If there exists i, j such that $\tilde{\sigma}_n^{ij}$ is contained in an $(\dim(\Sigma_n) - 1)$-dimensional cell τ of $\tilde{\mathcal{T}}_n$, using Lemma 4.9, we can consider a sufficiently small $\varepsilon > 0$ and a $\tilde{\mathcal{T}}_n^\varepsilon$ in which the carrier of $\tilde{\sigma}_n^{ij}$ in $\tilde{\mathcal{T}}_n^\varepsilon$ has dimension $\dim(\Sigma_n)$: this is done by choosing suitable perturbations $p(\varepsilon)$ of vertices p of the carrier of $\tilde{\sigma}_n^{ij}$ in $\tilde{\mathcal{T}}_n$, yielding the desired $\tilde{\mathcal{T}}_n^\varepsilon$. Moreover, for a sufficiently small ε, $\tilde{\mathcal{T}}_n^\varepsilon$ satisfies (1), (2), (3) and (4.1). Therefore, if necessary by considering a small perturbation $\tilde{\mathcal{T}}_n^\varepsilon$, we can assume that $\tilde{\mathcal{T}}_n$ satisfies (1), (2), (3) and (4.1).

Note now that the convex hull of any collection of $\dim(\Sigma_n) + \dim(\Sigma_{n+1})$ (or less) vertices of $\hat{T}_n^{ij} \times \hat{T}_{n+1}^{ij}$ is a union of convex sets whose dimensions are strictly less $\dim(\Sigma_n) + \dim(\Sigma_{n+1})$. This implies that if the vertices of $\hat{T}_n^{ij} \times \hat{T}_{n+1}^{ij}$ are positioned in $\tilde{T}_n^{ij}$ outside a set of dimension strictly less than $\dim(\Sigma_n) + \dim(\Sigma_{n+1})$, then $\tilde{\sigma}_n^{ij}$ satisfies property (5). By using Lemma 4.9 just like in the previous paragraph, we can then assume $\tilde{\mathcal{T}}_n$ satisfy (5).

To finish our construction it remains to prove that we can construct a piecewise linear convex function $\gamma_n : \Delta(\tilde{S}_{n,0}) \to \mathbb{R}_+$, linear precisely on each cell of $\tilde{\mathcal{T}}_n$ and constant over each $\hat{T}_n^{ij}$. Since $K_n[\dim(\Sigma_n) + 1] < m + 1$, the union of the set of vertices of $\hat{T}_n^{ij}$ over i, j spans an affine set with dimension lower than m. This implies we can define an affine function $\phi_n : \Delta(\tilde{S}_{n,0}) \to \mathbb{R}$ satisfying the following conditions: (a) $\phi_n(\hat{p}_n^{ij}) = \hat{\gamma}_n(\hat{p}_n^{ij})$, for all vertices $\hat{p}_n^{ij}$ of $\hat{T}_n^{ij}$; (b) $\phi_n(p_n) < \hat{\gamma}_n(p_n)$ for all remaining vertices p_n of $\tilde{\mathcal{T}}_n$. Letting now $\gamma_n \equiv \gamma_n' - \phi_n$ yields the desired function satisfying all the required properties in the statement of the Lemma. □

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